Bayesian Conditional Density Filtering for Big Data

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January 15, 2014

Abstract

We propose a Conditional Density Filtering (C-DF) algorithm for efficient online Bayesian inference. C-DF adapts Gibbs sampling to the online setting, sampling from approximations to conditional posterior distributions obtained by tracking of surrogate conditional sufficient statistics as new data arrive. This tracking eliminates the need to store or process the entire data set simultaneously. We show that C-DF samples converge to the exact posterior distribution asymptotically, as sampling proceeds and more data arrive over time. We provide several motivating examples, and consider an application to compressed factor regression for streaming data, illustrating competitive performance with batch algorithms that use all of the data.

Keywords: Approximate MCMC; Big data; Density filtering; Dimension reduction; Sequential Monte Carlo; Streaming data; Time series.

1 Introduction

In order to be useful in modern massive and streaming data settings, statistical methods must have computational implementations that scale efficiently in the number of samples \( n \), while exploiting low dimensional structure in the data in settings with large parameter spaces (e.g., regression in \( p \) predictors with \( p \) large).

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For big data, algorithms must be efficient in data storage and processing, avoiding bottlenecks in memory and number of operations required as the problem size (in both $n$ and $p$) increases.

Bayesian methods provide a natural probabilistic characterization of uncertainty in the parameters and in predictions, but there is a lack of scalable Bayesian inference algorithms having guarantees on accuracy. Sampling algorithms remain the gold standard, with the overwhelming emphasis being on Markov chain Monte Carlo (MCMC). State of the art performance is obtained in small to moderate sized problems, but MCMC fails to scale as the data and number of parameters grow. MCMC suffers from the need to update a large number of parameters with expensive likelihood evaluations that may involve $O(n^3)$ operations per iteration.

MCMC can be extended to accommodate data collected over time by adapting the transition kernel $K_t$, and drawing a few samples at time $t$, so that the samples converge in distribution to the joint posterior distribution $\pi_t$ as the time index $t$ increases (Yang and Dunson, 2013). Sequential MCMC requires the full data to be stored, leading to an increased storage and processing bottleneck as time increases and more data are accrued. If sufficient statistics are available, tracking these can alleviate some of the computational and storage burden, though issues still arise when the parameter space is large. A number of alternative strategies have been proposed for scaling MCMC to big data settings. One possibility is to parallelize computation within each MCMC iteration using GPUs or multicore architectures to free bottlenecks in updating unknowns specific to each sample and in calculating likelihoods (Medlar et al., 2013). Another possibility is to rely on Hamiltonian Monte Carlo with stochastic gradient methods used to approximate gradients with subsamples of the data (Welling and Teh, 2011). Ahn, Korattikara and Welling (2012) proposed a variation on this theme. Korattikara, Chen and Welling (2013) instead use sequential hypothesis testing to choose a subsample to use in approximating the acceptance ratio in Metropolis-Hastings.

In simple conjugate models, such as Gaussian state-space models, efficient updating equations can be obtained using methods related to the Kalman filter. Assumed density filtering (ADF) was proposed (Lauritzen, 1992; Boyen et al., 1998; Opper et al., 1999) to extend this computational tractability to broader classes of models. ADF approximates the posterior distribution with a simple conjugate family, leading to approximate online posterior tracking. The predominant concern with this approach is the propagation of errors with each additional approximation to the posterior in time. Expectation-propagation (EP) (Minka et al., 2009; Minka, 2013) improves on ADF through additional iterative refinements, but the approximation is limited to the class of assumed densities and has no convergence guarantees.
Sequential Monte Carlo (SMC) (Chopin, 2002; Arulampalam, 2008; Lopes et al., 2010) is a popular technique for online Bayesian inference that relies on resampling and propagating samples over time. Unfortunately, it is difficult to scale SMC to problems involving large $n$ and $p$ due to the need to employ very large numbers of particles to obtain adequate approximations and to particle degeneracy issues, addressed through expensive particle rejuvenation using all the data. Potentially, one can rejuvenate only at earlier time points, but for models involving many parameters this may not protect sufficiently against degeneracy.

There is a parallel literature on online variational approximations (Wang et al., 2011; Hoffman et al., 2010). These methods focus primarily on improving batch inferences by feeding in data sequentially. Broderick et al. (2013) recently proposed a streaming variational Bayes algorithm, while Hoffman et al. (2012) combined stochastic approximation with variational inference for better scaling. Relying on a factorized form of the posterior, variational methods are unable to capture true dependence in the joint posterior and severely underestimate uncertainty. There are no theoretical guarantees on accuracy except in simple specialized cases.

We propose a new class of Conditional Density Filtering (C-DF) algorithms that extend Gibbs sampling (Geman and Geman, 1984) to streaming data. Sampling in C-DF proceeds by drawing from conditional posterior distributions, but instead of conditioning on the conditional sufficient statistics (CSS) (Johannes et al., 2010; Carvalho et al., 2010), C-DF conditions on surrogate conditional sufficient statistics (SCSS), formed using sequential samples or point estimates for parameters along with the data. This leads to an approximation of the conditional distributions, which improves as data accrue over time and produces a provably good approximation to the target posterior even for moderate sample sizes. While the entire data or sufficient statistics need to be stored to update CSS, SCSS updating requires only data at the current time. In many settings (e.g. exponential family conditional distributions) tracking of SCSS is highly efficient, enabling online sampling of sets of parameters with linear time complexity with the arrival of new data. Relative to sufficient statistics, SCSS are lower-dimensional and can be calculated faster, while requiring less storage.

Section 2 presents a generic version of the C-DF algorithm, along with definitions, basic assumptions, and a template for how to derive desirable updating quantities. Section 3 demonstrates the versatility of the C-DF algorithm, applying it to a variety of popular regression contexts, and eliciting various advantages over competitors in each. Section 4 studies the assumptions of the C-DF algorithm and its convergence properties. Section 5 applies C-DF to a novel framework for supervised compressed regression, and Section
2 The C-DF algorithm

Define $\Theta = (\theta_1, \theta_2, \ldots, \theta_k)$ as the collection of unknown parameters in probability model $P(Y|\Theta)$ and $Y \in \mathcal{Y}$, with $\theta_j \in \Psi_j$, and $\Psi_j$ denoting an arbitrary sample space (e.g., a subset of $\mathbb{R}^n$). In the massive or streaming data context, $D_t$ is the data observed at time-point $t$, and $D^{(t)}$ denotes the entire data observed up to and including time $t$.

**Definition** Suppose $\theta_j | \theta_{-j}, D_t \overset{L}{=} \hat{\theta}_j | \theta_{-j,1}, h(D_t, \theta_{-j,2})$, with $\theta_{-j} = \Theta \setminus \theta_j$ and $\theta_{-j} = (\theta_{-j,1}, \theta_{-j,2})$. Then, $J^{(t)}$ is a conditional sufficient statistic (CSS) of $\theta_j$ if $\theta_j \perp D^{(t)} | \theta_{-j,1}, J^{(t)}$ at time $t$. Here, for known functions $f, h$, $J^{(t)} = f(h(D_1, \theta_{-j,2}), \ldots, h(D_t, \theta_{-j,2}))$ and $\theta_j | \theta_{-j}, D^{(t)} \overset{L}{=} \hat{\theta}_j | \theta_{-j,1}, J^{(t)}$.

By definition, CSS $J^{(t)}$ depends explicitly on $\theta_{-j,2}$ and thus changes whenever new samples are drawn for this collection of parameters. This necessitates storing entire data $D^{(t)}$ or some available sufficient statistics. In many problems, sufficient statistics often do not exist or scale poorly with the size of the parameter-space. Instead, we propose surrogate conditional sufficient statistics (SCSS) as defined below.

**Definition** Suppose $\theta_j | \theta_{-j}, D_t \overset{L}{=} \theta_j | \theta_{-j,1}, h(D_t, \theta_{-j,2})$, with $\theta_{-j} = \Theta \setminus \theta_j$ and $\theta_{-j} = (\theta_{-j,1}, \theta_{-j,2})$. Define $C^t = g(C^{t-1}, h(D_t, \hat{\theta}_{-j,2}^t))$ as the SCSS for $\theta_j$ with $\hat{\theta}_{-j,2}^t$ being a consistent estimator of $\theta_{-j,2}$ at time $t$ for some known functions $g$ and $h$. Then, $\theta_j | \theta_{-j}, D^{(t)}$ is approximated by $\theta_j | \theta_{-j,1}, C^t$, and $C^t$ is sufficient for drawing approximate samples of $\theta_j$ from the latter. Note: SCSS is not an approximation of CSS, but rather a means to approximate full conditional distributions of $\theta_j$. Further, any SS or CSS is also a SCSS, though the converse is not necessarily true.

We partition $\Theta$ into two disjoint sets indexed by $\mathcal{I}_1$ and $\mathcal{I}_2$, with parameters in each set assumed to admit surrogate conditional sufficient statistics. In general, the two sets are identified such that the SCSS for parameters in one set depend solely on consistent estimators of parameters in the other, along with the data through time $t$. The partitioning is context specific, and can often be performed in different ways. For massive data, these sets are identified to achieve large computational and storage gains. Let $s \in S = \{1, 2\}$, with $s' = S \backslash \{s\}$. Then, for each $j \in \mathcal{I}_s$, C-DF approximates full conditionals $\pi_t(\theta_j | \Theta_{-j}, D^{(t)})$ with $\tilde{\pi}_t(\theta_j | \Theta_{s(j)}, C^t_{j,s})$. Here, $\Theta_{s(j)} = \{\theta_l : l \in \mathcal{I}_s, l \neq j\}$, $\hat{\Theta}_{s'}$ denotes a consistent estimator for $\Theta_{s'} = \{\theta_j :
\( j \in I \) \( s \)'s using data up to time \( t \), and \( C_{js}^t \) denotes the SCSS for parameter \( \theta_j \) in class \( s \) at time \( t \). In this notation, \( \theta_{-j,1} = \Theta_s(j) \), \( \theta_{-j,2} = \Theta_{s'} \), and \( C_{js}^t = g(C_{js}^{t-1}, h(D_t, \hat{\Theta}_s^t)) \).

The C-DF algorithm alternates between sampling from approximate full conditionals for \( \theta_j : j \in I_1 \) and obtaining consistent estimators for \( \theta_j : j \in I_2 \), and vice versa. In doing so, C-DF obtains efficient updating equations to calculate \( C_{js}^t \) based on \( C_{js}^{t-1} \) and incoming data \( D_t \). This leads to an efficient method for parameter inference, with sampling-based approximations to the posterior or predictive distributions. As demonstrated in section 3, full conditional distributions often have SCSS or can be approximated by distributions with SCSS.

Several other scenarios can occur, including the following cases for which section 4 details supporting theory.

**Case 1** Parameters in both classes admit conjugate full conditional distributions. Approximate posterior draws are sampled from \( \tilde{\pi}_t(\theta_j | -) \) which has closed form.

**Case 2** Parameters in \( I_2 \) do not have conjugate full conditionals. Sample from \( \tilde{\pi}_t(\theta_j | -) \) using Metropolis Hastings. For high-dimensional parameters, an independence sampler may suffer from poor mixing, and one may consider more advanced MCMC methods (e.g. adaptive MCMC (Roberts and Rosenthal, 2009), Slice sampling (Neal, 2003), HMC (Duane et al., 1987), MALA (Roberts and Tweedie, 1996)), now adapted to use surrogate conditional sufficient statistics \( C_{j1}^t \).

**Case 3** Parameters in \( I_2 \) do not have conjugate full conditionals. Approximate non-conjugate \( \tilde{\pi}_t(\theta_j : I_2 | \Theta_{2(j)}, C_{j1}^t) \) with conditionally conjugate distribution \( q_t(\theta_j | \Theta_{2(j)}, \tilde{C}_{j1}^t) \), having approximate surrogate conditional sufficient statistics \( \tilde{C}_{j1}^t \) (e.g. Laplace approximation).

### 3 Motivating examples

The section illustrates the C-DF algorithm through several simple examples. We observe that when tracking of sufficient statistics is possible, C-DF and sequential MCMC (SMCMC) produce remarkably similar inference. Moreover, propagating surrogate conditional sufficient statistics using C-DF often results in significant computational and storage savings for online inference with high-dimensional data.

In the following case-studies, data subsets (shards) of fixed size \( n_t = n = 100 \) arrive sequentially over \( T = 1000 \) times. Where appropriate, \( X^t = (x_1^t, \ldots, x_n^t)' \) and \( y^t = (y_1^t, \ldots, y_n^t)' \) denote the batch of data.
Algorithm 1 Conditional Density Filtering (C-DF) Algorithm

\textbf{Model}: \(P(Y | \Theta), Y \in \mathcal{Y} \).

\textbf{Data Input}: Data \(D_t\) at time \(t\).

\textbf{Parameters}: \(\Theta = \Theta_1 \cup \Theta_2\) for disjoint index sets \(\mathcal{I}_1\) and \(\mathcal{I}_2\).

\textbf{Algorithm output}: Define \(S = \{1, 2\}, s \in S\), and \(s' = S \setminus \{s\}\), and \(\mathcal{M}_s^j = \{\Theta_{s(j)}, C_{js}^t\}\) for \(j \in \mathcal{I}_1 \cup \mathcal{I}_2\).

* Stored surrogate conditionally sufficient statistics (SCSS) \(C^t_j = \{C_j^t\}\), \(j \in \mathcal{I}_1 \cup \mathcal{I}_2\).
* Stored parameter estimates \(\hat{\theta}_j\) from \(\tilde{\pi}_t(\theta_j | \mathcal{M}_s^j)\) or \(q_t(\theta_j | \mathcal{M}_s^j)\).
* Approximate \(L\) posterior samples drawn as \(\theta_{j,t} \sim \tilde{\pi}_t(\theta_j | \mathcal{M}_s^j), j \in \mathcal{I}_1 \cup \mathcal{I}_2\).
* Approximate \(L\) predictive samples drawn as \(y^*_t \sim y^*(\theta_{\mathcal{R}}, D^{(t)}), \text{ with } \Theta_{\mathcal{R}} \subseteq \Theta\).

For each time \(t\):

1. For each \(j \in \mathcal{I}_1\):
   a. Update SCSS \(C^t_{j,1} = g(C_{j,1}^{t-1}, h(D_t, \tilde{\Theta}_2))\)
   b. Sample \(\theta_j\) from its conjugate approximate full conditional \(\tilde{\pi}_t(\theta_j | \mathcal{M}_s^j)\)

2. For each \(j \in \mathcal{I}_2\):
   a. If case 3: update approximate SCSS \(C^t_{j,2} = g(C_{j,2}^{t-1}, h(D_t, \tilde{\Theta}_1))\)
      Else: update SCSS \(C^t_{j,2} = g(C_{j,2}^{t-1}, h(D_t, \tilde{\Theta}_1))\)
   b. If case 3: sample \(\theta_j \sim q_t(\theta_j | -)\)
      Else: sample \(\theta_j \sim \tilde{\pi}_t(\theta_j | \mathcal{M}_s^j)\)

3. Obtain approximate online inference:
   a. Draw \(L\) samples from approximating distribution \(\tilde{\pi}_t(\theta_j | \mathcal{M}_s^j)\) or \(q_t(\theta_j | -)\) where appropriate
   b. Use approximate posterior draws to obtain approximate inference on \(y^*\) for a new observation \(x^*\)

observed at time \(t = 1, \cdots, T\). Define \(D^{(t)} = \{D_1, \cdots, D_t\}\) with \(D_h = (X^h, y^h)\), and similarly \(y^{(t)} = \{y^1, \cdots, y^t\}\). At each time 500 Gibbs iterations are drawn to approximate the corresponding posterior distribution \(\pi_t\). Figure 1 plots kernel-smoothed estimates of marginal posterior densities for representative parameters at various time points. Autocorrelation and mixing-rates of chains for C-DF and sequential MCMC were monitored and found to be virtually identical.

In example 3.2, ADF is added as an additional competitor, where the joint posterior over model parameters is approximated in time assuming a multivariate-normal class of densities. Transformations are made (e.g. to non-negative scale parameters) to ensure the joint parameter support is \((-\infty, \infty)^d\), with \(d\) being the dimension of parameter vector \(\theta\). Let \(\tilde{\pi}_t(\theta) \sim N_d(\mu_t, \Sigma_t)\), then \((\mu_t, \Sigma_t)\) is updated in the sense of McCormick et. al (2012). There, updating equations are given as \(\Sigma_t = -\nabla^2 \ell(\mu_{t-1})\) and \(\mu_t = \mu_{t-1} + \Sigma_{t-1} \nabla \ell(\mu_{t-1})\), with \(\ell(\theta) = \log(p(y | \theta) \tilde{\pi}(\theta))\). In both cases, ADF produces accurate parameter point-estimates, but severely underestimates uncertainty as shown in Figure 1(f).
3.1 Linear regression

We start with the linear regression example with

\[ y^i = X^i \beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I_n). \] (3.1)

With standard conjugate priors, \( \beta \sim N(0, \sigma^2 I_p), \sigma^2 \sim IG(a, b) \), the posterior for \( (\beta, \sigma^2) \) is

\[ \sigma^2 | D^{(t)} \sim IG(a', b'), \quad \text{with} \quad a' = a + tn, \quad b' = b + S^{YY}_t - S^{XY}_t(S^{XX}_t + I_p)^{-1}S^{XY}_t \]

\[ \beta | \sigma^2, D^{(t)} \sim N(\mu_t, \Sigma_t), \quad \text{with} \quad \Sigma_t = \sigma^2(S^{XX}_t + I_p)^{-1}, \quad \mu_t = \Sigma_t S^{XY}_t. \]

The parameters of the posterior distributions are defined in terms of sufficient statistics \( S^{XX}_t = S^{XX}_{t-1} + X^tX^t, \quad S^{YY}_t = S^{YY}_{t-1} + X^ty^t, \) and \( S^{XY}_t = S^{XY}_{t-1} + y^ty^t \), enabling online inference using sequential MCMC by propagating these quantities in time. In this simple setting, quantities for C-DF are trivial transformations of the sufficient statistics formed via the following steps: (1) set \( \sigma^2 = \hat{\sigma}^2_t \), the mode of the posterior distribution of \( \sigma^2 \) in time; (2) propagate sufficient statistics \( S^{XX}_t \) and \( S^{YY}_t \) (which are also conditional sufficient statistics), along with surrogate conditional sufficient statistic \( C^{XY}_t = \hat{C}^{XY}_t + \hat{\sigma}^2_t X^ty^t \). Draw samples for \( \beta \) from its approximate full conditional, namely \( \beta \sim N(\hat{\mu}_t, \hat{\Sigma}_t) \), with \( \hat{\Sigma}_t = \hat{\sigma}_t(S^{XX}_t + I_p)^{-1} \) and \( \hat{\mu}_t = (S^{XX}_t + I_p)^{-1}C^{XY}_t \).

The performance of SMCMC and C-DF are compared under a variety of initializations, with the objective of inference on \( (\beta, \sigma^2) \). Covariates are drawn iid from \( U(0, 1) \), and we let \( \beta_0 = (1, 0.5, 0.25, -1, -0.5) \) and \( \sigma^2_0 = 1 \). Kernel smoothed density estimates for both C-DF and SMCMC are plotted across a large number of time points. Figure 1 presents the density estimates at time 10 and 20, demonstrating C-DF and SMCMC produce nearly identical marginal inference even in small sample sizes.

3.2 One-way Anova

Now considering the one-way ANOVA model, we assume \( k \) groups with the number of replicates constant in each group. The model is

\[ y_{ij} = \zeta_i + \epsilon_{ij}, \quad \zeta_i \sim iid N(\mu, \tau^2), \quad \epsilon_{ij} \sim N(0, \sigma^2), \quad (3.2) \]
with $\pi(\mu) \propto 1$, $\tau^2 \sim IG(a, b)$, and $\sigma^2 \sim IG(\alpha, \beta)$. We consider a fixed number of treatment groups $(k)$, with a growing number of observations per group. The model admits sufficient statistics which can be propagated in time to yield an efficient SMCMC sampler. Let $y_i^t = (y_{i1}^t, \ldots, y_{im}^t)'$ denote the set of observations for the $i$-th group at time $t$ with $n = 10$. Sufficient statistic $S_i$ is updated as $S_i^t = S_i^{t-1} + \sum_{j=1}^n y_{ij}^t$ specific to each group, having observed a new data shard at time $t$. Online inference proceeds using SMCMC with a sequence of $m_t$ draws from the following Gibbs full conditionals:

$$
\zeta_i^t \sim N\left(\frac{\sum_{i=1}^k \zeta_i^t \sigma^2_i}{\sum_{i=1}^k \sigma^2_i}, \frac{\sigma^2}{\sum_{i=1}^k \sigma^2_i}\right), \mu_i^t \sim N\left(\frac{\sum_{i=1}^k \zeta_i^t \sigma^2_i}{\sum_{i=1}^k \sigma^2_i}, \frac{\sigma^2}{\sum_{i=1}^k \sigma^2_i}\right), \sigma^2_i \sim IG\left(\frac{\sum_{i=1}^k \zeta_i^t \sigma^2_i}{\sum_{i=1}^k \sigma^2_i}, \frac{\sigma^2}{\sum_{i=1}^k \sigma^2_i}\right).
$$

This updating scheme carries over to C-DF but set $\tau^2 = \tau^2_{t-1}$, the mode of its full conditional distribution fixing $\zeta_i$ and $\hat{\mu}$ as the final draw at time $(t - 1)$, and propagate surrogate conditional sufficient statistics $C_i^t = C_i^{t-1} + \hat{\tau}_i^2 S_i$ and $C_2^t = C_2^{t-1} + \sum_{i=1}^k (\hat{\mu}_i^t y_{i1}^t - 2\zeta_i^t S_i + n \zeta_i^2)^2$). Then sample

$$
\zeta_i^t \sim N\left(\frac{C_i^t + \sigma^2 \mu_i^t}{nt \hat{\tau}_i^2 + \sigma^2}, \frac{\hat{\tau}_i^2 \sigma^2}{nt \hat{\tau}_i^2 + \sigma^2}\right), \mu_i^t \sim N\left(\frac{\sum_{i=1}^k \zeta_i^t \sigma^2_i}{k}, \frac{\sigma^2}{k}\right), \sigma^2_i \sim IG\left(\frac{C_2^t}{2}, \frac{\hat{\tau}_i^2}{2}\right).
$$

In this example, there is no computational or storage benefit from using C-DF because sufficient statistics are univariate, but our goal here is to assess accuracy. We generate data using $\zeta_i \sim iid N(4, 0.01)$ with $\sigma^2 = 100$. The marginal posterior densities from C-DF and SMCMC for $\zeta_5$ at time 10 and 20 are shown in Figure 1. The posterior densities have remarkably similar peaks and spread, even for small sample sizes.

### 3.3 Normal Means Problem

In the preceding examples, there was minimal computational benefit of the C-DF algorithm. We now consider the more challenging normal means problem in which the number of parameters grows linearly with sample size and no sufficient statistics can be propagated. The response model is

$$
y_i^t = \theta_i^t + \epsilon_i^t, \quad \epsilon_i^t \sim N(0, 1), \quad \theta_i^t \sim N(\mu, \sigma_\theta^2),
$$

with $\mu \sim N(0, 1), \sigma_\theta^2 \sim IG(a, b), i = 1, \ldots, n_t$ denoting new data at time $t$, and $t = 1, \ldots, T$. We let $n_t = n$ for simplicity, $y^t = (y_1^t, \ldots, y_m^t)'$ and $\theta^t = (\theta_1^t, \ldots, \theta_n^t)'$. A typical goal is to infer $(\mu, \sigma^2)$ and obtain
the predictive distribution for the mean $\theta^*$ for a new observation,

$$
\pi(\theta^*|y^{(t)}) = \int N(\theta^*|\mu, \sigma^2_{\theta}) \pi(\mu, \sigma^2_{\theta}|y^{(t)}) d(\mu, \sigma^2_{\theta}).
$$

The conjugate Gibbs full conditionals at time $t$ are

$$
\mu \sim N\left(\frac{\sum_{h \leq t,j \leq n} \theta^h_{ij}}{nt + \sigma^2_{\theta}}, \frac{\sigma^2_{\theta}}{nt + \sigma^2_{\theta}}\right), \theta^h_i \sim N\left(\frac{\sigma^2_{\theta} y^h_i + \mu}{1 + \sigma^2_{\theta}}, \frac{\sigma^2_{\theta}}{1 + \sigma^2_{\theta}}\right), \sigma^2_{\theta} \sim IG\left(a + \frac{nt}{2}, b + \sum_{h \leq t,j \leq n} (\theta^h_i - \mu)^2\right).
$$

Using C-DF, full conditionals for $\mu$ and $\sigma^2_{\theta}$ admit SCSS as a function of $\hat{\Theta}$, eliminating the need to store the entire data vector, and bypassing the need to resample the entire collection of $\{\theta^h_i\}_{h=1}^t$. Assume surrogate conditional sufficient statistics $C^t_1$ and $C^t_2$ have been propagated through time $t$, then the algorithm proceeds as

1. Observe a new shard $y^{t+1} = (y_1^{t+1},...,y_n^{t+1})$ at time $t+1$. Fix $\theta^{t+1}_i = \hat{\theta}^{t+1}_i \sim \pi(\theta^{t+1}_i|\mu, \sigma^2_{\theta}, y^{t+1})$, where $\mu, \sigma^2_{\theta}$ are draws (consistent estimators) obtained at time $t$.

2. Update $C^{t+1}_1 = C^t_1 + \hat{\theta}^{t+1}_i$, and draw $\mu \sim N\left(\frac{C^{t+1}_1}{nt+1+\sigma^2_{\theta}}, \frac{\sigma^2_{\theta}}{nt+1+\sigma^2_{\theta}}\right)$.

3. Update $C^{t+1}_2 = C^t_2 + (\hat{\theta}^{(t+1)}_i)^2$, and draw $\sigma^2_{\theta} \sim IG\left(\frac{(t+1)}{2} + a, \frac{b_{\sigma^2_{\theta}}}{}\right)$, where $b_{\sigma^2_{\theta}} = C^{t+1}_2 - 2\mu C^{t+1}_1 + (t+1)\mu^2$.

The performance of C-DF in estimating posterior distributions for $(\mu, \sigma^2_{\theta})$ is assessed through the following simulation study: sample $\theta_i \sim N(\mu = 4, \sigma^2 = 0.25)$, and draw observations $y_i \sim N(\theta_i, 1)$. Figure 1 plots densities for $\mu$ and $\sigma$ for the approximation using C-DF and the Gibbs sampler on the entire dataset, for $T = \{5 \times 10^3, 10^4\}$. We obtain good posterior inference for our model parameters. This is a non-trivial result, considering the SCSS being tracked in this example are a function of a growing number of parameter estimates, including estimates for parameters which may be very crude initially. With no SS, the Gibbs sampler relies on storing the entire dataset, which is prohibitive for non-univariate responses and large time horizons. Further, resampling of $\Theta = (\theta^h_{ij})_{i=1,h=1}$ scales linearly in time.
Figure 1: Marginal posterior densities via C-DF vs. competitors for the motivation examples. True coefficients are show by vertical lines.
4 Convergence Behavior of Approximate Samplers

This section studies the convergence behavior for a general class of approximate MCMC algorithms, with the C-DF algorithm a special case. We first present a general result on the limiting error of approximation when a kernel with the targeted stationary distribution is approximated by another kernel. This result deals with finite samples, and is not limited to any specific class of approximations. We then show that such approximations improve with increasing sample size, yielding draws from the exact posterior asymptotically. Proofs are provided in an appendix.

4.1 Notation and Framework

Denote the data observed until time \( t \) by \( D^{(t)} = \{ D_1, \ldots, D_t \} \), and let \( \{ \pi_t(\cdot|D^{(t)}) : t \in \mathbb{N}_+ \} \) be the corresponding sequence of posterior distributions. \( \pi_t(\cdot|D^{(t)}) \) is sequence of probability measures defined on the sequence of measure spaces \( (H_t, \mathcal{H}_t), t \geq 1 \). In what follows, we consider \( H_t = \mathbb{R}^p \) and \( \mathcal{H}_t = \mathcal{B}(\mathbb{R}^p) \), the Borel \( \sigma \)-algebra on \( \mathbb{R}^p \) for a fixed \( p \)-dimensional parameter space \( \Theta = (\theta_1, \cdots, \theta_p) \). \( \pi_t \) admits density \( \pi_t(\Theta) \) with respect to the Lebesgue measure \( d\nu(\Theta) = d\nu_1(\theta_1), d(\theta_2), \cdots, d\nu_p(\theta_p) \). Let \( T_t : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^+ \) be the transition kernel which defines the parameter updating process at time \( t \), then

(1) \( T_t(x, \cdot) \) is a probability measure for all \( x \in \mathbb{R}^p \);

(2) \( T_t(\cdot, A) \) is a measurable function w.r.t the \( \sigma \)-algebra for all \( \nu \)-measurable \( A \).

A function \( f_t : \mathbb{R}^p \to \mathbb{R}^+ \) defined over \( \sigma \)-field \( \mathcal{H}_t \) is the invariant distribution of \( T_t \) if

\[
 f_t(\Theta') = \int T_t(\Theta, \Theta') f_t(\Theta) d\nu(\Theta). \tag{4.1}
\]

To denote the transition kernels and stationary distributions for finite sample cases, we omit the subscript \( t \). In studying convergence of a sequence of distributions, we focus on the total variation norm defined by

\[
 d_{TV}(\mu_1, \mu_2) = \sup_A |\mu_1(A) - \mu_2(A)|.
\]

4.2 A finite sample error bound for approximate MCMC samplers

We begin by characterizing how error propagates while approximating a kernel by another kernel.
Lemma 4.1 Let \( K \) be a kernel approximated by a kernel \( T \) s.t.

\[
\sup \| K(\Theta, \cdot) - T(\Theta, \cdot) \|_{TV} \leq \rho.
\]

Assume, \( \mu_1, \mu_2 \) are the stationary distributions of \( T \) and \( K \) respectively. Further assume, \( \| T^{(r)} - \mu_1 \|_{TV} \to 0 \) and \( \| K^{(r)} - \mu_2 \|_{TV} \to 0 \). Then \( \exists r_0 \) s.t.

\[
\sup \| K^{(r)}(\Theta, \cdot) - T^{(r)}(\Theta, \cdot) \|_{TV} \leq r\rho, \ \forall \ r \leq r_0
\]

and

\[
\sup \| K^{(r)}(\Theta, \cdot) - T^{(r)}(\Theta, \cdot) \|_{TV} \leq \min\{r\rho, 2\| \mu_1 - \mu_2 \|_{TV} \}, \ \forall \ r > r_0.
\] (4.2)

For small sample sizes with data observed at once or in a finite number of shards, the C-DF kernel will remain unchanged once SCSS for the last shard is calculated. Referring to Lemma 4.1, the approximation error due to C-DF increases initially before stabilizing. For streaming data, with certain additional assumptions C-DF can be shown to generate draws from the exact posterior distribution as more data accumulate.

4.3 Convergence results for a general approximation class

Define \( \Theta = (\Theta_1, \Theta_2) \), with \( \Theta_1 = (\theta_{11}, \ldots, \theta_{1p_1})' \in \mathcal{R}^{p_1}, \Theta_2 = (\theta_{21}, \ldots, \theta_{2p_2})' \in \mathcal{R}^{p_2}, \) and \( p_1 + p_2 = p \). Approximating kernel \( T \) for C-DF can be of many forms, depending on if \( \Theta_2 \) admits conjugate full conditional distributions. As outlined in section 2, there are three cases in particular which we concern ourselves with.

1. All full conditional distributions \( \theta_{1i} | \theta_{1,-i}, \hat{\Theta}_2, i = 1, \ldots, p_1 \) and \( \theta_{2i} | \theta_{2,-i}, \hat{\Theta}_1, i = 1, \ldots, p_2 \) are conjugate so that Gibbs sampling can be employed. Define approximate transition kernel \( T_t : \mathcal{R}^{p_1} \times \mathcal{R}^{p_2} \to \mathcal{R}^+ \) by

\[
T_t(\Theta, \Theta') = \left[ \prod_{i=1}^{p_1} \pi_t(\theta'_{1i} | \hat{\Theta}_{2,t-1}, \theta'_{1l}, l < i, \theta_{1l}, l > i) \right] \left[ \prod_{i=1}^{p_2} \pi_t(\theta'_{2i} | \hat{\Theta}_{1,t-1}, \theta'_{2l}, l < i, \theta_{2l}, l > i) \right],
\] (4.3)

where \( \{\hat{\Theta}_{1,t}\}_{t=1}^\infty, \{\hat{\Theta}_{2,t}\}_{t=1}^\infty \) are two sequences of estimators.

2. All full conditional distributions \( \theta_{1i} | \theta_{1,-i}, \hat{\Theta}_2, i = 1, \ldots, p_1 \) exist in conjugate form, however, \( \theta_{2i} | \theta_{2,-i}, \hat{\Theta}_1, i = 1, \ldots, p_2 \) do not admit closed form conditional distributions. Here, C-DF proceeds in one of two
Lemma 4.2 In both case (1) and (2.1), \( f \) stationary distribution \( \pi \) Assume \( f \)

\[ T_t(\Theta, \Theta') = \prod_{i=1}^{p_1} \pi_t(\theta'_{1i} | \hat{\Theta}_2, t-1, \theta'_{1l}, l < i, \theta_{1l}, l > i) Q(\Theta_2, \Theta'_2 | \hat{\Theta}_1, t-1), \]  

(4.4)

where \( \{\hat{\Theta}_1, t\}_{t=1}^\infty \), \( \{\hat{\Theta}_2, t\}_{t=1}^\infty \) are two sequences of estimators.

(2.2) Approximate full conditional distributions by full conditional distributions of \( q_t \) which allows propagation of conditional sufficient statistics. The approximate kernel is given by

\[ T_t(\Theta, \Theta') = \prod_{i=1}^{p_1} \pi_t(\theta'_{1i} | \hat{\Theta}_2, t-1, \theta'_{1l}, l < i, \theta_{1l}, l > i) q_t(\Theta_2, \Theta'_2 | \hat{\Theta}_1, t-1). \]  

(4.5)

Assume \( f_t : \mathcal{R}^p \rightarrow \mathcal{R}^+ \) is the invariant distribution of \( T_t \). The following Lemma specifies the unique stationary distribution \( f_t \) of \( T_t \) for each of these cases outlined.

**Lemma 4.2** In both case (1) and (2.1), \( f_t(\Theta) = \pi_t(\Theta_1 | \hat{\Theta}_2, t-1) \pi_t(\Theta_2 | \hat{\Theta}_1, t-1) \).

**Remark:** Using a similar proof as in Lemma 4.2 it is easy to show the stationary distribution for \( T_t \) in case 2.2 is \( \bar{f}_t(\Theta) = \pi_t(\Theta_1 | \hat{\Theta}_2) \pi_t^q(\Theta_2 | \hat{\Theta}_1) \); here, \( \pi_t^q(\cdot) \) denotes the stationary distribution of approximate transition kernel \( q_t \).

Let \( \pi_0 \) be the initial distribution from which parameters are drawn. Below we state the main result.

**Theorem 4.3** Assume, (i) \( \exists \alpha_\ell \in (0, 1) \) s.t. \( \forall t, \sup \Theta d_{TV}(T_t(\Theta, \cdot), f_t) \leq 2\alpha_\ell \), (ii) \( d_{TV}(f_t, f_{t-1}) \rightarrow 0 \), and (iii) \( d_{TV}(f_t, \pi_t) \rightarrow 0 \), then \( \exists \{n_t\}_{t \geq 1} \) s.t. \( d_{TV}(T_t^{(n_t)} \cdots T_1^{(n_1)} \pi_0, \pi_t) \rightarrow 0 \).

For case 2.2 we need one additional assumption for the result to go through.

**Corollary 4.4** Assume (i) \( \exists \alpha_\ell \in (0, 1) \) s.t. \( \forall t, \sup \Theta d_{TV}(T_t(\Theta, \cdot), \tilde{f}_t) \leq 2\alpha_\ell \), (ii) \( d_{TV}(\tilde{f}_t, \tilde{f}_{t-1}) \rightarrow 0 \), (iii) \( d_{TV}(\tilde{f}_t, f_t) \rightarrow 0 \), and (iv) \( d_{TV}(f_t, \pi_t) \rightarrow 0 \). then \( \exists \{n_t\}_{t \geq 1} \) s.t. \( d_{TV}(T_t^{(n_t)} \cdots T_1^{(n_1)} \pi_0, \pi_t) \rightarrow 0 \).

The proof of Corollary 4.4 follows from Theorem 4.3 and hence is omitted.

**Remark 1:** In essence, Theorem 4.3 states that running a Markov chain with approximate kernel \( T_t \) for \( n_t \) times at each time \( t \) will asymptotically have draws from the true joint posterior distribution.
Remark 2: Condition (i) in Theorem 4.3 is referred to as the universal ergodicity condition (Yang & Dunson, 2013). It is shown in Lemma 3.2 in Yang & Dunson (2013) that the universal ergodicity condition is weaker than uniform ergodicity condition on the transition kernel \( T \). Condition (ii) ensures that the stationary distribution of the approximating kernel changes slowly as time progresses. Lemma 3.7 in Yang & Dunson (2013) shows that condition (ii) is satisfied for any regular parametric model by applying a simple Bernstein-Von Mises theorem. Finally, the stationary distribution of the approximating chain is required to be “close” to the true posterior distribution at later time; this is formalized in condition (iii). Sufficient conditions under which assumption (iii) holds are outlined in Lemma 4.5. Before stating this Lemma, we recall the definition of posterior consistency.

Definition: A posterior \( \Pi(\cdot \mid D^{(t)}) \) is defined to be consistent at \( \Theta^0 \) if, for every neighborhood \( U \) of \( \Theta^0 \), \( \Pi(U \mid D^t) \rightarrow 1 \) under the data generating law at \( \Theta^0 \).

Lemma 4.5 Assume that the likelihood function \( p_\Theta(\cdot) \) is continuous as a function of \( \Theta \) at \( \Theta^0 = (\Theta_1^0, \Theta_2^0) \) and \( \sqrt{t} p_\Theta(D^t) \) in limit is bounded away from 0 and \( \infty \). Suppose \( \Theta^0 \) is an interior point in the domain, with the prior distribution \( \pi_0(\Theta_1, \Theta_2) \) being positive and continuous at \( \Theta^0 \). Further, assume \( \hat{\Theta}_{1,t} \rightarrow \Theta_1^0 \), \( \hat{\Theta}_{2,t} \rightarrow \Theta_2^0 \) a.s. under the data generating law at \( \Theta_0 \), and \( f_t \) and \( \pi_t \) are both consistent at \( \Theta^0 \). Then

\[
\int |\pi_t(\Theta) - f_t(\Theta)| d\Theta \rightarrow 0 \text{ as } t \rightarrow \infty,
\]

almost surely under the true data generating model at \( \Theta^0 \).

5 Supervised Compressed Regression

5.1 Model description and implementation

We now consider a more challenging application to compressed linear regression,

\[
y_t \mid \Phi, \beta, \sigma^2 \sim N(X_t \Phi' \beta, \sigma^2 I_n),
\]

where \( y_1, \ldots, y_t \in \mathbb{R}^n \) are \( n \)-dimensional response vectors observed over time with associated features \( X_1, \ldots, X_t \in \mathbb{R}^{n \times p} \), with \( \Phi \) an \( m \times p \) matrix with \( m \ll p \). As \( \Phi' \beta = \Phi' LL' \beta \), for any orthogonal
Matrix \( L, \Phi \) and \( \beta \) are only identified up to an orthogonal transformation. However, one can identify the coefficients \( \gamma = \Phi' \beta \). We let \( (\beta, \sigma^2) \sim N(0, \sigma^2 \Sigma_\beta) \times IG(a, b) \), obtaining Jeffrey’s prior by letting \( a, b \to 0 \). In addition, we choose a matrix-normal prior, \( \Phi \sim MN(\Phi_0, K, 1_m) \), centered on a row orthonormalized random projection matrix, with \( K = \text{diag}(\kappa_1, \ldots, \kappa_m) \) containing row scalings on the prior variance. Each row of \( \Phi \) is assumed a priori independent with \( \kappa_i \overset{iid}{\sim} IG(c/2, d/2) \) for \( i = 1, 2, \ldots, m \).

Recursive updates for sufficient statistics up to time \( t \) are given by \( F_{ty}^t = F_{ty}^{t-1} + y_t' y_t, F_{xy}^t = F_{xy}^{t-1} + y_t' X_t, \) and \( F_{xx}^t = F_{xx}^{t-1} + X_t' X_t \). Conditional on \( \Phi \), marginal posterior distributions for \( \beta, \sigma^2 \) are

\[
\beta | \Phi, D^{(t)} \sim T_n(\mu_t, \Sigma_t), \quad \sigma^2 | \Phi, D^{(t)} \sim IG(a_{1,t}/2, b_{1,t}/2),
\]

where \( T_n(\cdot) \) is the multivariate-\( t \) distribution with \( \nu \) degrees of freedom. With \( W = \Phi F_{xx}^t \Phi' + \Sigma_\beta^{-1} \), the hyper parameters in (5.2) are given as

\[
a_{1,t} = n t, \quad b_{1,t} = F_{ty}^t - F_{xy}^t \Phi' W^{-1} \Phi F_{xy}^t, \quad \Sigma_t = b_{1,t}/n W^{-1}, \quad \mu_t = (b_{1,t}/n)^{-1} \Sigma_t \Phi F_{xy}^t.
\]

The conjugate update for \( \Phi = [\Phi_1, \Phi_2, \ldots, \Phi_p] \) proceeds column wise, with a Gibbs within Gibbs step. For column \( j \), define \( z_{jt} = y_t - \sum_{t \neq k} X_{kt} \Phi_k' \beta \), with \( X_{kt} \) as the \( l \)-th column of \( X_t \), then

\[
\Phi_j | \{ \Phi \}_{-j}, K, \beta, D^{(t)} \sim N_m(\mu_{\Phi_j}, \Sigma_{\Phi_j}),
\]

\[
\Sigma_{\Phi_j} = \left( \sum_{s=1}^t \beta X_{js}' X_{js} / \sigma^2 + K^{-1} \right)^{-1}, \quad \mu_{\Phi_j} = \Sigma_{\Phi_j} \left( \sum_{s=1}^t \beta X_{js}' z_{js} / \sigma_s^2 + K^{-1} \Phi_{0j} \right).
\]

Finally update \( \kappa_i \) according to the full conditional \( \kappa_i | \Phi \sim IG(c_i'/2, d_i'/2) \), with \( c_i = c + p \) and \( d_i = d + (\Phi(i,.) - \Phi_0(i,.))' (\Phi(i,.) - \Phi_0(i,.)) \). Here, \( \Phi(i,.) \) represents the \( i \)-th row of \( \Phi \). An SMCMC implementation (Yang and Dunson, 2013) proceeds as:

1. At each time point \( t \), update sufficient statistics \( F_{xy}^t, F_{xx}^t, \) and \( F_{ty}^t \).

2. Draw a number of MCMC samples from the full conditional distributions at time \( t \).

3. Observe new data \( D_{t+1} = \{ X_{t+1}, y_{t+1} \} \) at time \( t + 1 \) and go to (1).

SMCMC faces poor mixing in the high-dimensional joint parameter space \( \beta, \Phi, \sigma^2 \), compounded by
correlation between $\beta$ and $\Phi$, requiring a large number of samples to obtain a reasonable effective sample size. The requirements for storing sufficient statistics grow quadratically with $p$, becoming burdensome in high dimensional problems. Further, updating these quantities has complexity $O(np^2) + O(np)$, a bottleneck for efficient online inference. We were also unable to obtain efficient inferences using conventional MCMC repeated at each time or SMC.

These problems can be addressed using C-DF. Following Algorithm 1, the model parameters are split into $\Theta_{x1} = (\beta, \sigma^2)$ and $\Theta_{x2} = (\Phi, K)$, and approximate inference proceeds as:

1. Observe data $D_t = \{X_t, y_t\}$ at time $t$. Update SCSS $CF_t^{XX} = CF_{t-1}^{XX} + \Phi_{t-1} X'_t \hat{X}_{t-1}, CF_t^{XY} = CF_{t-1}^{XY} + \Phi_{t-1} X'_t y_t$, and $CF_t^{yy'} = CF_{t-1}^{yy'} + y'_t y_t$.

2. Set $\hat{\sigma}_t^2 = \hat{\sigma}_{MAP}^2 = b_{1,t}/(a_{1,t} + 1)$. Following notation in (5.3), define $W = CF_t^{XX} + \Sigma_{\beta}^{-1}$, where $a_{1,t} = nt, b_{1,t} = CF_t^{yy} - CF_t^{XY} W^{-1} CF_t^{XY'}$.

3. Set $\hat{\beta}_t = \hat{\beta}_{MAP} = \mu_t$, where $\Sigma_t = b_{1,t}/n W^{-1}$, and $\mu_t = (b_{1,t}/n)^{-1} \Sigma_t CF_t^{XY'}$.

4. Draw $L$ samples for approximate posterior inference using (5.2), but now with the hyper parameters as defined above.

5. Draw $L$ samples for $\Phi$ from $\Phi|K, D^{(l)} \propto p(\Phi|K) \prod_{t=1}^t p(y_t|\Phi, \hat{\beta}_t, \hat{\sigma}_t^2)$:

   5.1. Full conditional $\Phi|K, D^{(l)}, K|\Phi, D^{(l)}$ are conjugate and admit SCSS; sampling from the former is intractable due to the need to invert an $mp \times mp$ matrix. Following (5.4), sample $\Phi$ column-wise; these full conditionals also admit SCSS.

   5.2. Update SCSS $SF_{t,j}^{XX} = SF_{t-1,j}^{XX} + (\hat{\beta}_t X'_t)(X'_{j} \hat{X}_{t}), SF_{t,j}^{XY} = SF_{t-1,j}^{XY} + \hat{\beta}_t X'_t y'_{j}$.

   5.3. Sample each column $\Phi_j \sim [\Phi_j|-\] with $\Sigma_{\Phi_j} = (\sum_{s=1}^t SF_{t,j}^{XX}/\sigma^2 + K^{-1})^{-1}$ and $\mu_{\Phi_j} = \Sigma_{\Phi_j} (SF_{t,j}^{XY}/\hat{\sigma}_t^2 + K^{-1} \Phi_{0j})$.

   5.4. Run the nested sampler for $100 + L$ iterations, with a 100 iteration burn-in. Set $\hat{\Phi}_t$ as the average over the final $L$ draws.

   5.5. Store the final $L$ samples as approximate draws for $\Phi$ at stage $t$.

6. For approximate posterior draws for $\gamma$, compute $\gamma_{ti} = \Phi'_t \beta_{ti}$ for $i = 1, 2, \ldots, L$. 

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Updating SCSS $\mathbf{CF}_t^{XX}$, $\mathbf{CF}_t^{Xy}$ for C-DF has complexity $O(mp_n) + O(m^2 n)$ and $O(mp_n) + O(m n)$, respectively. $\mathbf{SF}_{t,j}^{XX}$, $\mathbf{SF}_{t,j}^{Xy}$'s are low dimensional and trivial to store and propagate. By reducing the quadratic dependence on $p$ to the linear order, C-DF massively reduces the computational time resulting in fast online inference. Also, regularizing the conditioning set with robust estimators, C-DF massively improves mixing and convergence in the Markov chain. The following section assesses performance in simulation examples.

### 5.2 Simulation setup

To test effectiveness of C-DF in compressed regression, we perform simulation experiments in various settings. We consider cases in which $\gamma$ is sparse in (5.1) and also when $\beta$ is sparse but not $\gamma$. In both cases, studies are performed to see how varying degrees of correlation and feature dimension affect performance of the C-DF algorithm.

The following quantities are fixed in all cases: number of new data points at each time ($n = 100$), number of time points ($N_s = 1000$), dimension of the subspace which $\Phi$ maps into ($m = 10$), and noise variance ($\sigma^2 = 4$). The following quantities are varied: (a) Number of features ($p = 500$ or $1000$); (b) Correlation ($\rho = 0.10$ or $0.40$) between the features. Each observation $x_i \sim N_p(0, \mathbf{R})$, with $R_{jk} = \rho^{|j-k|}$, for $i = 1, 2, \ldots, n$ and $j, k = 1, 2, \ldots, p$; (c) Signal strength (‘high’ or ‘low’): High corresponds to $\beta_j \sim \text{Un}(-3, 3)$ for every nonzero feature. Low corresponds to $\beta_j = 0.10$ for every nonzero feature.

<table>
<thead>
<tr>
<th>Case</th>
<th>Corr.</th>
<th>dim($p$)</th>
<th>$#\beta_j \neq 0$</th>
<th>Signal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>500</td>
<td>10</td>
<td>high</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>1000</td>
<td>10</td>
<td>high</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>500</td>
<td>10</td>
<td>high</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>1000</td>
<td>10</td>
<td>high</td>
</tr>
<tr>
<td>5</td>
<td>0.1</td>
<td>500</td>
<td>500</td>
<td>high</td>
</tr>
<tr>
<td>6</td>
<td>0.1</td>
<td>500</td>
<td>500</td>
<td>low</td>
</tr>
</tbody>
</table>

Table 1: Simulation cases for supervised compressed regression for streaming data.

The cases in Table 1 are not trained on massive data, but with sufficient sample size to illustrate C-DF’s scalability and performance relative to competing algorithms (Figure 2, Tables 2 and 3). 50 test replications were generated for each experiment in Table 1 to access predictive MSE, coverage, and length of posterior intervals. Bayesian shrinkage methods including BL (Park et al., 2008) and GDP (Armagan et al., 2013) were attempted, but standard implementations of these procedures require a $p \times p$ matrix inversion at each MCMC iteration, rendering them computationally infeasible. As an example, Bayesian Lasso was run using
the optimized blasso package in R for a batch dataset with \( n = 20,000 \) and \( p = 500 \). A 1,000 iteration MCMC analysis had a run-time of over 6 hours.

For model (5.1), sufficient statistics of the data can be propagated. Thus a sequential MCMC implementation is our baseline competitor to C-DF. Like C-DF, the chain in sequential MCMC is allowed to run for 30 iterations at each time point. SMC was also attempted, but this approach is very computationally inefficient due to severe particle degeneracy for learning high-dimensional parameter \( \Phi \), requiring frequent rejuvenation steps. Sufficient statistics for these updates are available, but do no better than a sequential MCMC implementation, while suffering the need store and propagate a large number of high-dimensional particles.

As a more scalable competitor, we derive a variational Bayes (VB) approximation to the posteriors obtained using a GDP prior on the coefficients of a standard linear regression model, \( y_i = x_i' \beta + \epsilon, \) for \( \epsilon \sim N(0, \sigma^2) \). Here \( \beta_j \mid \sigma \sim GDP(\zeta = \frac{\sigma^2}{\alpha}, \alpha) \) is equivalent to hierarchical prior \( \beta_j \mid \sigma, \tau_j \sim N(0, \sigma^2 \tau_j) \), with \( \tau_j \sim Exp(\lambda_j^2/2) \) and \( \lambda_j \sim Ga(\alpha, \eta) \). For \( \Theta = (\beta, \tau, \lambda, \sigma^2)' \), \( \tau = (\tau_1, \ldots, \tau_p)' \), and \( \lambda = (\lambda_1, \ldots, \lambda_p)' \), the mean-field VB approximation replaces exact posterior \( \pi(\Theta \mid D) \) with \( q(\Theta) = \prod q_j(\theta_j) \). For a density of this form, it can be shown that the optimal \( q_j(\theta_j) \) minimizing the KL distance between \( \pi(\theta \mid D) \) and \( q(\Theta) \) is \( q_j(\theta_j) \propto \exp \left[ E_{-q_j(\theta_j)} \{ \log \pi(\Theta, D) \} \right] \), where \( E_{-q(\theta)} \) denotes the expectation over distribution \( \prod_{i \neq j} q_i(\theta_i) \) and \( \pi(\Theta, D) = p(D \mid \Theta) \pi(\Theta) \). For more details see Ormerod & Wand (2010), Luts & Ormerod (2013).

5.3 Simulation results

The C-DF implementation of 5.1 produces excellent estimation and selection of the important variables as shown in Figure 2. In all simulation cases, C-DF has zero false positives and is able to recover the embedded lower dimensional structure even with increasing correlation between features. Credible intervals (not shown) for all variables are tight around true coefficient values. As \( p \) becomes very large (10,000+), coverage of the true values for predictor coefficients deteriorate, though C-DF still manages to identify important predictors. We also found robustness to the choice of \( m \), the dimension of the row-space of projection matrix \( \Phi \), as shown in Figure 5.3. Hence, we fixed \( m = 10 \) as a default in all cases; making \( m \) very large can result in over-fitting and C-DF scales as \( m^3 \) in the worst case, providing a computational reason to keep the dimension small.

Predictive MSE for each of the simulation settings averaged over 50 simulated datasets is shown in
Figure 2: Mean estimates of feature coefficients from C-DF and LASSO for final 100 time points. Figure 2(a), Figure 2(b), Figure 2(c) and Figure 2(d) show cases 1, 2, 3 and 4, respectively. The horizontal scale is taken as the log of the feature index.

Table 2. Subscripted values represent bootstrap standard errors for the averaged MSPEs, calculated by generating 100 bootstrap datasets resampled from the 50 MSPE values, finding the average MSPE of each,
and then computing their standard error. For the sparse simulated cases of Table 1, VB-GDP yields the lowest MSPE, though C-DF and sequential MCMC demonstrate competitive performance. Increasing the number of predictors causes MSPE to increase for all methods, as does an increase of correlation between predictors. C-DF performs well in all cases, while VB-GDP suffers for dense truths, and especially when the signal is small.

<table>
<thead>
<tr>
<th>Method</th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
<th>case 4</th>
<th>case 5</th>
<th>case 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>VB</td>
<td>3.430.005</td>
<td>4.200.006</td>
<td>3.490.005</td>
<td>4.230.006</td>
<td>3.520.007</td>
<td>8.790.010</td>
</tr>
</tbody>
</table>

Table 2: MSPE comparisons for the simulation cases of Table 1.

Table 3 shows the coverage probabilities of 95% predictive intervals. In all cases, C-DF has good predictive coverage while VB-GDP suffers drastically as expected for VB methods. For cases 1-4 of Table 1, the median lengths of the 95% PIs for C-DF are 9.58, 11.06, 9.72, 11.19, while VB-GDP produces overly narrow intervals.

In addition to the dramatic reduction in storage requirements, we have found the propagation of SCSS results in an efficiency gain over SMCMC; here, efficiency is measured as the time point $m_t$ at which MSPE
Table 3: Predictive coverage of all the competitors are shown for six different simulation studies. Empirical 95% confidence intervals of the coverage probabilities over 50 simulated datasets are shown in brackets.

drops and stays below a chosen threshold, here taken to be 5.0. The number of MCMC samples required to satisfy this criteria is defined as the algorithm’s “effective sample-size” score (smaller is better). Table 4 presents effective sample-sizes for C-DF and sequential MCMC in each of the outlined simulated cases averaged over 10 independent datasets. Given that both methods have a nearly identical per iteration runtime, C-DF’s significantly smaller effective sample-size implies that this algorithm results in a better model fit faster.

<table>
<thead>
<tr>
<th>Method</th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
<th>case 4</th>
<th>case 5</th>
<th>case 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>VB</td>
<td>0.82</td>
<td>0.83</td>
<td>0.82</td>
<td>0.83</td>
<td>0.81</td>
<td>0.83</td>
</tr>
<tr>
<td></td>
<td>(0.81,0.84)</td>
<td>(0.76,0.87)</td>
<td>(0.81,0.85)</td>
<td>(0.75,0.87)</td>
<td>(0.76,0.86)</td>
<td>(0.80,0.85)</td>
</tr>
<tr>
<td>C-DF</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.97</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>(0.98,0.99)</td>
<td>(0.97,0.99)</td>
<td>(0.98,0.99)</td>
<td>(0.97,0.98)</td>
<td>(0.97,1)</td>
<td>(0.99,1)</td>
</tr>
<tr>
<td>sequential MCMC</td>
<td>0.96</td>
<td>0.93</td>
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<td>0.96</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td>(0.95,0.96)</td>
<td>(0.91,0.96)</td>
<td>(0.95,0.96)</td>
<td>(0.90,0.96)</td>
<td>(0.96,0.96)</td>
<td>(0.93,0.97)</td>
</tr>
</tbody>
</table>

Table 4: Effective sample-size to achieve MSPE threshold in each simulation study.

### 6 Real data illustrations

This section demonstrates C-DF in two F16 aircraft flying control applications. An aileron is a hinged control surface attached to the trailing edge of the wings, and is primarily used to control the aircraft in roll, or movement around the aircraft’s longitudinal axis. For the “Ailerons” data, the goal is to predict the control action of the ailerons based on attributes related to the status of the airplane (e.g. climb-rate), while in the “Elevators” data, we predict the aircraft’s angle of elevation. Details can be found at [http://www.dcc.fc.up.pt/~ltorgo/Regression/DataSets.html](http://www.dcc.fc.up.pt/~ltorgo/Regression/DataSets.html). In addition, there is interest in identifying important features. We analyze the data as in the previous section.

VB, sequential MCMC and BCR (Guhaniyogi and Dunson, 2013) are chosen as competitors. While BCR and sequential MCMC are performed sequentially by feeding in blocks of data, VB-GDP is run on
the entire data in batch mode. Although our proposed method is applicable to huge datasets, the examples shown here deal with large datasets with moderate number of predictors. The nature of these data make them likely to exhibit sparsity in the relationship between the response and predictors.

**Ailerons Data** The data are divided into 143 time points, each with \( n = 50 \) observations and \( p = 40 \) features. The test set has 6595 observations. Several columns have identical values, so we added white-noise to each of these entries prior to standardization. MSPE for C-DF is competitive to VB, BCR and sequential MCMC, stabilizing very quickly to \( \approx 0.57 \) within the first 10 time points. Consistent with the simulation studies, lengths of predictive intervals and predictive coverage are comparable across all methods in these two real data settings except for VB (see Table 5), for which a severe under-coverage takes place. VB-GDP selects 18 out of 40 variables, while C-DF selects 33 variables. In all cases, the coefficients of non-selected features are found to be small. BCR also performs competitively, attributed to this dataset having relatively weak signal and low sparsity.

**Elevators Data** In this dataset 8750 observations in the training set were split randomly into 50 time points, each with \( n = 175 \) observations and \( p = 14 \) features. The signal strength in this dataset appears somewhat higher compared to the Ailerons data. C-DF has the best performance among the competitors, while BCR has the worst performance. Predictive coverage and interval lengths were similar for all methods but VB, which again shows poor coverage.

<table>
<thead>
<tr>
<th>Performance metrics</th>
</tr>
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<tbody>
<tr>
<td>Method</td>
</tr>
<tr>
<td>D1 VB</td>
</tr>
<tr>
<td>BCR</td>
</tr>
<tr>
<td>C-DF</td>
</tr>
<tr>
<td>sequential MCMC</td>
</tr>
<tr>
<td>D2 VB</td>
</tr>
<tr>
<td>BCR</td>
</tr>
<tr>
<td>C-DF</td>
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<tr>
<td>sequential MCMC</td>
</tr>
</tbody>
</table>

Table 5: Performance comparisons for real data illustrations after training on the full data. D1 and D2 are the Ailerons and Elevators data, respectively. #Zero refers to the number of features identified by the model as unrelated to the response.
7 Discussion

Conditional density filtering is a novel algorithm designed to scale Bayesian inference to massive data when it is distributed across a computer network or observed as a stream over time. C-DF uses the Gibbs sampling mechanism of drawing samples from conditional distributions, but makes an approximation by conditioning on sequential point estimates of certain other parameters instead of their samples. Sequential point estimation can rely on stochastic gradient descent, leading to significant computational savings relative to usual Gibbs. We have observed that C-DF maintains a good characterization of uncertainty, in sharp contrast to variational algorithms. In addition, in hard problems such as the compressed regression example, additional computational gains are obtained via substantial improvements in mixing relative to Gibbs; indeed in compressed regression Gibbs was essentially unusable due to poor mixing and scaling. C-DF’s good performance in approximating the posterior is supported by strong convergence results showing the asymptotic equivalence between approximate draws of C-DF and the true posterior distribution. We have focused in this article on modest-sized examples, which are challenging enough to lead to substantial problems with usual MCMC algorithms in order to demonstrate advantages. However, exploiting distributed architectures, C-DF can be easily applied in much larger problems involving large sample sizes and numbers of parameters.

8 Appendix

Proof of Lemma 4.1

The proof follows by induction. First we will prove an identity that will be used in the proof that follows. Letting $A \in B(\mathcal{R}^d)$,

$$K^{(r)}(\Theta, A) - T^{(r)}(\Theta, A) = \int \left[ K^{(r-1)}(\Theta', A) - T^{(r-1)}(\Theta', A) \right] T(\Theta, d\Theta')$$

$$+ \int \left[ K(\Theta, d\Theta') - T(\Theta, d\Theta') \right] K^{(r-1)}(\Theta', A). \quad (8.1)$$
(8.1) relates the differences between the kernels at $r$-th iteration of the Markov chain. Using the fact that r.h.s is free of $A$ and the relationship that $||\nu_1 - \nu_2||_{TV} = \sup_{g:R^d \to [0,1]} |\int g d\nu_1 - \int g d\nu_2|$, (8.1) yields

$$||K^{(r)}(\Theta, \cdot) - T^{(r)}(\Theta, \cdot)||_{TV} \leq ||K(\Theta, \cdot) - T(\Theta, \cdot)||_{TV}$$

$$+ \int ||K^{(r-1)}(\Theta', \cdot) - T^{(r-1)}(\Theta', \cdot)||_{TV} T(\Theta, d\Theta'). \quad (8.2)$$

Suppose (4.2) holds for $(r-1)$. Using (8.2) we find

$$\sup_{\Theta} ||K^{(r)}(\Theta, \cdot) - T^{(r)}(\Theta, \cdot)||_{TV} \leq \sup_{\Theta} ||K(\Theta, \cdot) - T(\Theta, \cdot)||_{TV}$$

$$+ \sup_{\Theta} \int ||K^{(r-1)}(\Theta', \cdot) - T^{(r-1)}(\Theta', \cdot)||_{TV} T(\Theta, d\Theta')$$

$$\leq \sup_{\Theta} ||K(\Theta, \cdot) - T(\Theta, \cdot)||_{TV} + (r-1)\rho < \rho + (r-1)\rho = \rho r. \quad (8.3)$$

Also note that $\exists r_0$ s.t. for all $r > r_0$, $||T^{(r)} - \mu_1||_{TV} < \frac{1}{2}||\mu_1 - \mu_2||_{TV}$ and $||K^{(r)} - \mu_2||_{TV} < \frac{1}{2}||\mu_1 - \mu_2||_{TV}$. Using triangle inequality, for all $r > r_0$

$$||K^{(r)} - T^{(r)}||_{TV} \leq ||K^{(r)} - \mu_2||_{TV} + ||\mu_1 - \mu_2||_{TV} + ||T^{(r)} - \mu_1||_{TV} < 2||\mu_1 - \mu_2||_{TV}. \quad (8.4)$$

Comparing (8.3) and (8.4) the result follows.

**Proof of Lemma 4.2**

We will show that $\int T_i(\Theta, \Theta') f_i(\Theta) d\Theta = f_i(\Theta')$. Note that in case (i)

$$L.H.S = \int \left[ \prod_{l=1}^{d_1} \pi_l(\Theta'_{1l} | \Theta_{2l-1}, \Theta_{1l}, l < i, \Theta_{1l}, l > i) \right] \left[ \prod_{l=1}^{d_2} \pi_l(\Theta'_{2l} | \Theta_{1l-1}, \Theta_{2l}, l < i, \Theta_{2l}, l > i) \right]$$

$$\pi_l(\Theta_{1l} | \Theta_{2l-1}) \pi_l(\Theta_{2l} | \Theta_{1l-1})$$

$$= \left( \int \prod_{l=1}^{d_1} \pi_l(\Theta'_{1l} | \Theta_{2l-1}, \Theta'_{1l}, l < i, \Theta_{1l}, l > i) \right) \pi(\Theta_{1l} | \Theta_{1l-1}) d\Theta_{1l}$$

$$\left( \int \prod_{l=1}^{d_2} \pi_l(\Theta'_{2l} | \Theta_{1l-1}, \Theta_{2l}, l < i, \Theta_{2l}, l > i) \right) \pi_l(\Theta_{2l} | \Theta_{1l-1}) d\Theta_{2l}$$

$$= \pi_l(\Theta'_{1l} | \Theta_{2l-1}) \pi(\Theta_{2l} | \Theta_{1l-1}).$$
The last step follows from the well known fact that \( \prod_{i=1}^{d_i} \pi_t(\Theta_{2i}^l|\Theta_{1l}^t, l \leq i, \Theta_{2l}^t, l > i) \).
\[
\prod_{i=1}^{d_i} \pi_t(\Theta_{1i}^l|\Theta_{1l}^t-1, \Theta_{1l}^t, l < i, \Theta_{1l}^t, l > i)
\]
are the Gibbs sampling kernels with the stationary distribution \( \pi_t(\Theta_{2l}^t|\Theta_{1l}^t,t-1) \) and \( \pi_t(\Theta_{1l}^t|\Theta_{2l}^t,t-1) \) respectively. The proof for case (ii) follows in an identical manner taking into account that the MH kernel \( Q(\Theta_{2l}^t, \Theta_{1l}^t|\Theta_{1l}^t,t-1) \) has \( \pi_t(\Theta_{2l}^t|\Theta_{1l}^t,t-1) \) as its stationary distribution.

**Proof of theorem 4.3**

The following proof builds on results from Yang & Dunson (2013). Although most of the proof coincides with the proof of Theorem 3.6 in Yang & Dunson (2013), for the sake of completeness we present the entire proof. Fix \( \epsilon \in (0, 1) \). Choose \( n_t, t \geq 1 \) s.t. \( \rho^{n_t} < \epsilon \). Using the fact that universal ergodicity condition implies uniform ergodicity, one obtains

\[
d_{TV}(T_{n_t}^{n_t}, \pi_t) \leq \alpha_t^{n_t} < \epsilon.
\]

Let \( h = T_{n_t-1}^{n_t} \cdots T_1^{n_1} \pi_0 \), then

\[
d_{TV}(T_{n_t}^{n_t} \cdots T_1^{n_1} \pi_0, f_t) = d_{TV}(T_{n_t}^{n_t} h, f_t) \leq d_{TV}(T_{n_t}^{n_t}, f_t) d_{TV}(h, f_t)
\]
\[
\leq \alpha_t^{n_t} d_{TV}(h, f_t) \leq \epsilon (d_{TV}(h, f_{t-1}) + d_{TV}(f_t, f_{t-1})). \tag{8.5}
\]

using the result repeatedly, one obtains

\[
d_{TV}(T_{n_t}^{n_t} \cdots T_1^{n_1} \pi_0, f_t) \leq \sum_{l=1}^{t} \epsilon^{t+1-l} d_{TV}(f_t, f_{l-1}).
\]

R.H.S clearly converges to 0 applying condition (ii). Finally, the proof is completed by using condition (iii) and the fact that

\[
d_{TV}(T_{n_t}^{n_t} \cdots T_1^{n_1} \pi_0, \pi_t) \leq d_{TV}(T_{n_t}^{n_t} \cdots T_1^{n_1} \pi_0, f_t) + d_{TV}(f_t, \pi_t).
\]
Proof of lemma 4.5

Note that the approximated posterior distribution \( f_t \) is given by

\[
f_t(\Theta_1, \Theta_2) = \pi_t(\Theta_1 | \hat{\Theta}_{1,t}) \pi_t(\Theta_2 | \hat{\Theta}_{1,t}) = \frac{\prod_{l=1}^t p_{\Theta_1, \hat{\Theta}_{2,t}}(D_{l,t}) \prod_{l=1}^t p_{\Theta_1, \hat{\Theta}_{2,t}}(D_{l,t}) \pi_0(\hat{\Theta}_{1,t}, \Theta_2) \pi_0(\Theta_1, \hat{\Theta}_{2,t})}{\int \prod_{l=1}^t p_{\Theta_1, \hat{\Theta}_{2,t}}(D_{l,t}) \prod_{l=1}^t p_{\Theta_1, \hat{\Theta}_{2,t}}(D_{l,t}) \pi_0(\hat{\Theta}_{1,t}, \Theta_2) \pi_0(\Theta_1, \hat{\Theta}_{2,t})}.
\]

Given that \( \hat{\Theta}_{1,t} \rightarrow \Theta^0_1, \hat{\Theta}_{2,t} \rightarrow \Theta^0_2 \) a.s. under \( \Theta_0 \), there exists \( \Omega_0 \) which has prob. 1 under the data generating law such that \( \forall \omega \in \Theta^0_0, \hat{\Theta}_{1,t}(\omega) \) and \( \hat{\Theta}_{2,t}(\omega) \) are in the arbitrary small neighborhood of \( \Theta^0_1 \) and \( \Theta^0_2 \) respectively.

As \( \pi_0 \) is cont. at \( \Theta^0 \), given \( \epsilon > 0 \) and \( \eta > 0 \) \( \exists \) a neighborhood \( N_{\epsilon, \eta} \) s.t. \( \forall \Theta \in N_{\epsilon, \eta} \) one has

\[
|\pi_0(\Theta_1, \Theta_2) - \pi_0(\Theta^0_1, \Theta^0_2)| < \epsilon. \tag{8.6}
\]

Using (8.6) and consistency of \( \hat{\Theta}_{1,t} \) and \( \hat{\Theta}_{2,t} \) one obtains, for all \( t > t_0 \) and \( \omega \in \Omega_0 \)

\[
|\pi_0(\Theta_1, \hat{\Theta}_{2,t}) - \pi_0(\Theta^0_1)| < \epsilon, |\pi_0(\hat{\Theta}_{1,t}, \Theta_2) - \pi_0(\Theta^0_2)| < \epsilon. \tag{8.7}
\]

Continuity of \( p_{\Theta}(\cdot) \) at \( \Theta^0 \) leads to the condition that for all \( t > t_0 \),

\[
|p_{\Theta_1, \Theta_2}(D_{l,t}) - p_{\Theta^0_1, \Theta^0_2}(D_{l,t})| < \epsilon. \tag{8.8}
\]

Also consistency assumptions on \( f_t \) and \( \pi_t \) yield that for all \( t > t_1 \) and \( \omega \in \Omega_1 \)

\[
f_t(N_{\epsilon, \eta}|D^{(t)}(\omega)) > 1 - \eta, \quad \pi_t(N_{\epsilon, \eta}|D^{(t)}(\omega)) > 1 - \eta,
\]

where \( \Omega_1 \) has probability 1 under the data generating law. Considering \( \Omega = \Omega_0 \cap \Omega_1 \) and \( t_2 = \max\{t_1, t_0\} \) it is evident that \( \Omega \) has also prob. 1 under the true data generating law and all of the above conditions hold for \( t > t_2 \) and \( \omega \in \Omega \).
Using similar calculations we have

\[
{f_t(\Theta | D^{(t)}(\omega)) \over \pi_t(\Theta | D^{(t)}(\omega))} = {f_t(N_{\epsilon,\eta} | D^{(t)}(\omega)) \over \pi_t(N_{\epsilon,\eta} | D^{(t)}(\omega))} 
\times \left[ \prod_{l=1}^t \frac{p_{\Theta_1,\Theta_2}(D_l)}{p_{\Theta_1,\Theta_2}(D_l)} \prod_{l=1}^t \frac{p_{\Theta_1,\Theta_2}(D_l)}{p_{\Theta_1,\Theta_2}(D_l)} \prod_{l=1}^t \frac{p_{\Theta_1,\Theta_2}(D_l)}{p_{\Theta_1,\Theta_2}(D_l)} \pi_0(\Theta_1,\Theta_2) \pi_0(\Theta_1,\Theta_2) \right] \left[ \frac{\int_{N_{\epsilon,\eta}} \prod_{l=1}^t p_{\Theta}(D_l) \pi_0(\Theta)}{\prod_{l=1}^t p_{\Theta}(D_l) \pi_0(\Theta)} \right]
\]

(8.9)

Using (8.7) we have

\[
(p_0(\Theta^0) - \epsilon)^2 \int_{N_{\epsilon,\eta}} \prod_{l=1}^t p_{\Theta_1,\Theta_2}(D_l) p_{\Theta_1,\Theta_2}(D_l)
\]

\[
\leq \int_{N_{\epsilon,\eta}} \prod_{l=1}^t \left[ p_{\Theta_1,\Theta_2}(D_l) p_{\Theta_1,\Theta_2}(D_l) \right] \pi_0(\Theta_1,\Theta_2) \pi_0(\Theta_1,\Theta_2)
\]

\[
\leq (p_0(\Theta^0) + \epsilon)^2 \int_{N_{\epsilon,\eta}} \prod_{l=1}^t p_{\Theta_1,\Theta_2}(D_l) p_{\Theta_1,\Theta_2}(D_l).
\]

Similarly,

\[
(p_0(\Theta^0) - \epsilon) \int_{N_{\epsilon,\eta}} \prod_{l=1}^t p_{\Theta}(D_l) \leq \int_{N_{\epsilon,\eta}} \left[ \prod_{l=1}^t p_{\Theta}(D_l) \right] \pi_0(\Theta) \leq (p_0(\Theta^0) + \epsilon) \int_{N_{\epsilon,\eta}} \prod_{l=1}^t p_{\Theta}(D_l).
\]

Therefore,

\[
{f_t(\Theta | D^{(t)}(\omega)) \over \pi_t(\Theta | D^{(t)}(\omega))} \leq (1 - \eta)^{-1} \left[ \prod_{l=1}^t \frac{p_{\Theta_1,\Theta_2}(D_l)}{p_{\Theta_1,\Theta_2}(D_l)} \right] \left[ \frac{\int_{N_{\epsilon,\eta}} \prod_{l=1}^t p_{\Theta}(D_l)}{\prod_{l=1}^t p_{\Theta}(D_l)} \right] \left[ \frac{(p_0(\Theta^0) + \epsilon)^3}{(p_0(\Theta^0) - \epsilon)^3} \right].
\]

Using similar calculations we have

\[
{f_t(\Theta | D^{(t)}(\omega)) \over \pi_t(\Theta | D^{(t)}(\omega))} \geq (1 - \eta) \left[ \prod_{l=1}^t \frac{p_{\Theta_1,\Theta_2}(D_l)}{p_{\Theta_1,\Theta_2}(D_l)} \right] \left[ \frac{\int_{N_{\epsilon,\eta}} \prod_{l=1}^t p_{\Theta}(D_l)}{\prod_{l=1}^t p_{\Theta}(D_l)} \right] \left[ \frac{(p_0(\Theta^0) - \epsilon)^3}{(p_0(\Theta^0) + \epsilon)^3} \right].
\]
Using the condition that $\sqrt{t}p_{\Theta_0}(D_t)$ as $t \to \infty$ is bounded away from 0 and $\infty$ and choosing $\epsilon, \eta$ small we have

$$\left| \frac{f_t(\Theta|D_t(\omega))}{\pi_t(\Theta|D_t(\omega))} - 1 \right| < \kappa,$$

for all $t > t_2$, $\omega \in \Omega$. Finally,

$$\int |\pi_t(\Theta) - f_t(\Theta)| \leq \int_{N_{\epsilon,\eta}} |\pi_t(\Theta) - f_t(\Theta)| + \int_{N_{\epsilon,\eta}} |\pi_t(\Theta) - f_t(\Theta)|$$

$$\leq \int_{N_{\epsilon,\eta}} |\pi_t(\Theta) - f_t(\Theta)| + 2\eta$$

$$\leq \pi_t(N_{\epsilon,\eta})\kappa + 2\eta < \kappa + 2\eta.$$

Hence proved.

References


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