BAYESIAN COMPUTATIONS FOR RELIABILITY GROWTH MODELING

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ABSTRACT

In this paper reliability growth models for both the attribute and variable data are considered. Bayesian analysis of these models requires inference over ordered regions. Even though closed form results for posterior quantities can be obtained in some cases [see for example, Mazzuchi and Soyer (1993)], in general when the number of test stages gets large computations become burdensome and more importantly the results become inaccurate due to the problems in evaluation of gamma functions involving the computations. In such cases, the posterior and predictive analyses can be done more efficiently using Markov Chain Monte Carlo (MCMC) methods. We discuss use of the MCMC methods for inference in both attribute and variable data reliability growth models and discuss extension of our approach to different reliability growth scenarios. We illustrate the implementation of the approach by using reliability growth data.

Key words: Bayesian inference, Markov chain Monte Carlo methods, ordered Dirichlet distribution, Gibbs sampling.
1. INTRODUCTION AND OVERVIEW

During the development phase of a new system or a product such as a piece of software, testing is performed in stages and at the end of each test stage, design changes/modifications are made to the system with the intention of improving its performance. This test/modify process, often termed "reliability growth", has been the focus of much attention in the reliability literature. It is common to classify the reliability growth models by the type of data testing yields, that is, attribute or variable.

The attribute testing scenario has been considered in the literature from both the sample theoretic [see for example Lloyd and Lipov (1962), Barlow and Scheuer (1966), Finkelstein (1983) and Johnson (1991)] and Bayesian [see for example Pollock (1968), Smith (1977), Mazzuchi and Soyer (1992), and Mazzuchi and Soyer (1993)] perspectives. Most models for variable data scenario involves use of point processes, more specifically the nonhomogeneous Poisson processes. See for example, Crow (1982) for sampling theory and Kyparis and Singpurwalla (1984) for Bayesian approaches. Other reliability growth models which are not based on nonhomogeneous Poisson process models include McWilliams (1979) who assumes a monotonic (nondecreasing) structure on the mean failure times associated with each stage of testing and uses isotonic regression methods for estimation.

In this paper we will consider reliability growth models for both the attribute and variable data scenarios and obtain Bayesian inferences for current stage reliabilities after each test stage as well as the future stage reliabilities. In addressing issues such as when to terminate the test/modify process and release the system, the future stage reliabilities are required in evaluating preposterior losses [see van Dorp, Mazzuchi, and Soyer (1994) for optimal stopping rules in reliability growth testing]. In dealing with attribute data we will consider the reliability growth model of Mazzuchi and Soyer (1993) which provides a unification of the previous Bayesian approaches to reliability growth modeling. For the case of variable data we present an extension of the Mazzuchi-Soyer model. Bayesian analysis of these models requires inference over ordered regions. Even though closed form results for posterior quantities can be obtained in the attribute data case, in general when the number of test stages gets large, the accuracy of the results becomes questionable due to the problems in evaluation of gamma functions. We show that in such cases, the posterior and predictive inferences can be obtained more efficiently and accurately using Markov Chain Monte Carlo (MCMC) methods. Use of MCMC methods facilitates also the evaluation of expected loss (utility) functions for determining optimal stopping rules in reliability growth tests.

In Section 2, we will review the attribute reliability growth model introduced in Mazzuchi and Soyer (1993) and illustrate the difficulties in computing the posterior and predictive inferences. We will also introduce an extension of this model to the variable data case where the failure rate of the system is assumed to be constant during a test stage and discuss the difficulties in inference. In Section 3, we will present inference for the two models using MCMC methods. Our approach here is based on a Gibbs-Importance sampling algorithm as in Mueller (1992). In Section 4, we discuss the extension of our approach to the Barlow-Scheuer model and a Weibull reliability growth model. In Section 5, we will illustrate the applications of our approach both to attribute and variable data from reliability growth tests.

2. BAYESIAN RELIABILITY GROWTH MODELS

2.1 A Reliability Growth Model for Attribute Data.

We assume a testing — modification scenario is conducted so that identical replications of the product are tested until a failure is observed. Upon the discovery of a failure, a modification
is made to the product to remove the cause and therefore increases product reliability. The
test—modification scenario is repeated for some specified number of times. Let \( m \) denote
the total number of test/modify stages, \( R_i, i = 1, \ldots, m \), denote the product reliability for
the \( i \)th stage of testing (that is, prior to the \( i \)th product modification), and \( R_{m+1} \) denote
the final (field) reliability. Because engineering modifications are made to the item upon
the discovery of a failure, it is reasonable to assume that

\[
0 \leq R_1 \leq \cdots \leq R_{m+1} \leq 1.
\]  (2.1)

Following Mazzuchi and Soyer (1993), a natural and mathematically tractable prior
distribution for \( R = (R_1, \ldots, R_{m+1}) \) is the ordered Dirichlet distribution given as

\[
\Pi(R | D^{(0)}) = \frac{\Gamma(\beta)}{\prod_{j=1}^{m+2} \Gamma(\beta \alpha_i)} \prod_{j=1}^{m+2} (R_j - R_{j-1})^{\beta \alpha_i - 1},
\]  (2.2)

where \( R_0 \equiv 0, R_{m+2} \equiv 1 \), and \( D^{(0)} \) represents the information prior to any testing as captured
by the prior parameters \( \beta, \alpha_i > 0, \sum_{i=1}^{m+2} \alpha_i = 1 \). We note that the distribution is defined over the
simplex \( \{R | 0 \leq R_1 \leq \cdots \leq R_{m+1} \leq 1\} \) and thus embodies the restrictions in (2.1) and
imposes no additional restrictions in the analysis.

An attractive feature of the ordered Dirichlet distribution is that all relevant marginal and
conditional distributions are Beta distributions. For example, if we define \( \alpha_i^* = \sum_{j=1}^{i} \alpha_j \), then it can
be shown that the marginal distributions are given by

\[
[R_i | D^{(0)}] \sim \text{Beta}
\left(\beta \alpha_i^*, \beta(1 - \alpha_i^*)\right),
\]  (2.3)

where \( \text{E}[R_i | D^{(0)}] = \alpha_i^* \), and \( \beta \) is a degree of belief parameter with lower values of \( \beta \) reflecting
more spread in the distribution. Similarly, we can show that

\[
[R_j - R_i | D^{(0)}] \sim \text{Beta}
\left(\beta(\alpha_j^* - \alpha_i^*), \beta(1 - \alpha_j^* + \alpha_i^*)\right), \text{ for } i < j,
\]  (2.4)

implying that \( \text{E}[R_j - R_{i-1} | D^{(0)}] = \alpha_i \) and the parameters \( \alpha_i \) can be interpreted as the expected
one step improvement in reliability from stage \( i - 1 \) to \( i \). Furthermore, we can show that the conditional distributions

\[
\Pi(R_i | R_{(-i)}, D^{(0)}) = \Pi(R_i | R_{i-1}, R_{i+1}, D^{(0)}),
\]  (2.5)

where \( R_{(-i)} = \{R_j | j \neq i\} \) and

\[
[R_i | R_{i-1}, R_{i+1}, D^{(0)}] \sim \text{Beta}
\left(\beta \alpha_i, \beta \alpha_{i+1}(R_{i-1}, R_{i+1})\right)
\]  (2.6)

denoting that the conditional distribution of \( R_i \) is a truncated beta density over \( (R_{i-1}, R_{i+1}) \).

The observation (sampling) model for the number of items tested in each stage \( i \), \( N_i \),
follows a geometric distribution of the form

\[
\Pr\left\{N_i = n_i | R_i\right\} = (1 - R_i)R_i^{n_i-1}, \quad n_i = 1, 2, \ldots,
\]  (2.7)

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and after the specification of the prior parameters, probability statements about $N_i$ for any stage $i$, can be made via the predictive distribution

$$
\text{Pr}\{N_i = n_i \mid D^{(0)}\} = \int_0^1 \text{Pr}\{N_i = n_i \mid R_i\} \Pi(R_i \mid D^{(0)}) dR_i = \frac{\beta(1 - \alpha_i^*) \prod_{j=0}^{n_i-2} (\beta \alpha_i^* + j)}{\prod_{j=0}^{n_i-1} (\beta + j)} \quad (2.8)
$$

for $n_i = 1, 2, \ldots$, where $\prod_{j=0}^{\ast} \{ \ast \} \equiv 1$. It follows from the above that

$$
E[N_i \mid D^{(0)}] = E[E[N_i \mid D^{(0)}, R_i]] = \frac{\beta - 1}{\beta(1 - \alpha_i^*) - 1} \quad (2.9)
$$

for $\beta > 1/(1 - \alpha_i^*)$.

Defining $D^{(i)} \equiv \{D^{(0)}, n_1, \ldots, n_i\}$, the likelihood function of $R_j$ after $i$ stages of testing is given by $L(R_j \mid D^{(i)}) = \prod_{j=1}^i (1 - R_j) R_j^{n_j-1}$ and the posterior distribution of $R_j$ is obtained as

$$
\Pi(R_j \mid D^{(i)}) \propto \left\{ \prod_{j=1}^i (1 - R_j) R_j^{n_j-1} (R_j - R_{j-1})^{\beta a_{j-1}} \right\} \left\{ \prod_{j=i+1}^{m+2} (R_j - R_{j-1})^{\beta a_j-1} \right\} \quad (2.10)
$$

After the $i$th testing stage, the interest will be in the quantities $R_1, \ldots, R_{m+1}$, the current and future product reliabilities. We therefore integrate out $(R_1, \ldots, R_{i-1})$ over the region $0 \leq R_1 \leq \cdots \leq R_{i-1} \leq R_i$. As shown in van Dorp, Mazzuchi and Soyer (1994), after considerable amount of algebraic manipulations, $\Pi(R_1, \ldots, R_{m+1} \mid D^{(i)})$ can be obtained as a mixture of ordered Dirichlet distributions of the form as in (2.2) and thus all prior distributional characteristics are preserved in the posterior as mixtures. Thus, given $D^{(i)}$, the stage reliabilities can be predicted by taking weighted averages of the prior expressions evaluated for revised parameter values. But when $m$ is large (say, $m \geq 10$), the evaluation of the weights becomes difficult and the results become inaccurate due to the problems in evaluation of gamma functions. As it will be discussed in the Section 3, these computations can be done more efficiently and accurately by using MCMC methods.

2.2 An Extension of the Reliability Growth Model for Variable Data.

Let $X_i, i=1, 2, \ldots$, denote the life-length of the system during the $i$-th stage of testing and $\lambda_i$ denote the failure rate of the system during the $i$-th stage. If we assume that the system does not exhibit any aging, then given $\lambda_i$ the failure behavior of the system is described by the exponential failure model with density

$$
\text{p}(x_i \mid \lambda_i) = \lambda_i \exp(-\lambda_i x_i) \quad (2.11)
$$

Similar to the attribute reliability growth model case of Section 2.1, we will assume that the identical copies of the system will be tested until a failure is observed and upon the discovery of the failure, a modification is made to remove the cause to improve the system's reliability. Each copy will be tested until the failure or at most $\tau$ units of time. As before we assume that the
test — modification process is repeated for m stages and we let \( \lambda_{m+1} \) denote the system failure rate in the field. Under the reliability growth scenario we expect that \( \{\lambda_i\} \) is a decreasing sequence in i. To reflect this we will define \( R_i = \exp(-\lambda_i) \) and note that \( 0 < R_i < 1 \) for all i. If we assume that the \( R_i \)'s are defined over the simplex \( \{ R_i \mid 0 \leq R_1 \leq \cdots \leq R_{m+1} \leq 1 \} \) as in Section 2.1 then the failure rates will be restricted as

\[
\infty > \lambda_1 > \lambda_2 \cdots > \lambda_{m+1} > 0. \tag{2.12}
\]

Following the development in Section 2.1, we assume that the prior on \( R \) is an ordered Dirichlet given by (2.2). We note that once we define the prior on \( R_i \)'s we have the prior on \( \lambda_i \)'s defined over the simplex (2.12) due to the one-to-one transformation \( R_i = \exp(-\lambda_i) \). For example,

\[
\Pi[\lambda_i \mid D^{(0)}] \propto e^{-\lambda_i(\beta \alpha^*_i + 1)} \left(1 - e^{-\lambda_i}\right)^{\beta(1-\alpha^*_i)}, \tag{2.13}
\]

where \( \alpha^*_i \) and \( D^{(0)} \) are defined as in Section (2.1). Predictive density of \( X_i \)

\[
p(x_i \mid D^{(0)}) = \int_0^\infty p(x_i \mid \lambda_i) \Pi(\lambda_i \mid D^{(0)}) d\lambda_i \tag{2.14}
\]

can not be analytically obtained in this case. We will refer to the above model as the exponential reliability growth model.

The likelihood of \( \lambda_j \), given \((n_j - 1) \) survivals and a failure, at the \( j \)-th testing stage is

\[
\mathcal{L}(\lambda_j; x_j, n_j) = \lambda_j \exp(-\lambda_j x_j) \exp[-\lambda_j T(n_j - 1)]. \tag{2.15}
\]

Defining \( D^{(l)} = \{D^{(0)}, x_1, n_1, x_2, n_2, \cdots, x_l, n_l\} \), the posterior distribution of \( R \) can be obtained as

\[
\Pi(R \mid D^{(l)}) \propto \left( \prod_{j=1}^{m+2} (R_j - R_{j-1})^\alpha q^{-1} \right) \left( \prod_{j=1}^i \ln(1/R_j) R_j^{x_j + r(n_j - 1)} \right). \tag{2.16}
\]

In this case the posterior distribution \( \Pi(R_1, \cdots, R_{m+1} \mid D^{(0)}) \) and therefore \( \Pi(\lambda_1, \cdots, \lambda_{m+1} \mid D^{(0)}) \), cannot be obtained in any type of analytical form since we can not integrate out (2.16) with respect to \((R_1, \cdots, R_{i-1}) \) over the region \( 0 \leq R_1 \leq \cdots \leq R_{i-1} \leq R_i \). Thus, the only way to make inference in this case is to use the MCMC methods as will be discussed in Section 3.

### 3. BAYESIAN INFERENCE VIA MONTE CARLO METHODS

In the sequel we will give an overview of the Gibbs sampler for making posterior and predictive inference in the reliability growth models of Section 2. The attractive nature of the Gibbs sampler is that it enables us to generate samples from the posterior distributions in an indirect manner without having to obtain the corresponding distributions. For a more detailed discussion of the Gibbs sampler and other related Monte Carlo methods, see Gelfand and Smith (1990) and Casella and George (1992).
3.1 An Overview of Gibbs Sampler.

Let \( \varrho = (\varrho_1, \varrho_2, \ldots, \varrho_m) \) denote some unknown quantities, such as the reliabilities at different stages of testing. We are interested in the joint distribution of \( \Pi(\varrho) \), as well as the marginal distributions of individual \( \theta_i \)'s. We can think of \( \Pi(\varrho) \) as the posterior distribution of \( \varrho \) with the dependence on data is suppressed. The Gibbs sampler enables to draw samples from \( \Pi(\varrho) \) without actually computing the distribution. This is achieved by successive drawings from the full conditional distributions \( \Pi(\theta_i | \varrho^{(-i)}) \) where \( \varrho^{(-i)} = \{\theta_j | j \neq i \} \) for \( i = 1, 2, \ldots, m \).

We start the process with a vector of arbitrary starting values

\[
\varrho^0 = (\varrho_1^0, \varrho_2^0, \ldots, \varrho_m^0),
\]

and

- draw \( \theta_1^1 \) from \( \Pi(\theta_1 | \varrho_2^0, \ldots, \varrho_m^0) \)
- draw \( \theta_2^1 \) from \( \Pi(\theta_2 | \theta_1^1, \varrho_3^0, \ldots, \varrho_m^0) \)
- draw \( \theta_3^1 \) from \( \Pi(\theta_3 | \theta_1^1, \theta_2^1, \varrho_4^0, \ldots, \varrho_m^0) \)

\[\vdots\]
- draw \( \theta_m^1 \) from \( \Pi(\theta_m | \theta_1^1, \theta_2^1, \ldots, \theta_{m-1}^1) \).

As a result of the single iteration of the Gibbs sampler in (3.1), we have generated a single vector which represents a transition from the starting value \( \varrho^0 = (\varrho_1^0, \varrho_2^0, \ldots, \varrho_m^0) \) to \( \varrho^1 = (\theta_1^1, \theta_2^1, \ldots, \theta_m^1) \). If we repeat this iteration \( k \) times, then we generate the Gibbs sequence

\[
\varrho^0, \varrho^1, \varrho^2, \ldots, \varrho^k,
\]

which is a realization of a Markov Chain. Under some mild regularity conditions, distribution of \( \varrho^k \) converges to the posterior distribution \( \Pi(\varrho) \) as \( k \to \infty \) and \( \varrho^k \) gives us a sample from \( \Pi(\varrho) \). To able to generate a sample from \( \Pi(\varrho) \), then, one alternative is to generate \( s \) independent Gibbs sequences of \( k \) iterations each and use the \( k \)-th value from each sequence as a sample point from \( \Pi(\varrho) \). An alternate way is to generate a single long Gibbs sequence and after disregarding an initial burn-in sample, to collect every \( l \)-th value from the remaining sequence to form a sample from \( \Pi(\varrho) \) [see Smith and Roberts (1993) for a detailed discussion of other implementation issues].

Once we form a sample \( \varrho^1, \varrho^2, \ldots, \varrho^r \) from the posterior distribution \( \Pi(\varrho) \), the marginal posterior distributions of \( \theta_i \)'s can be obtained from the sample points \( \theta_1^1, \theta_2^2, \ldots, \theta_r^r \). Alternatively, if the full conditional distribution for \( \theta_i \) is available, the marginal posterior distribution can be obtained by using

\[
\Pi(\theta_i) \approx \frac{1}{r} \sum_{j=1}^{r} \Pi(\theta_i | \varrho^{(j-1)}) .
\]

The predictive density of an observable \( X \) can also be obtained as
\[ p(x) = \int p(x|\theta_j) \Pi(\theta_j) \, d\theta_j \approx \frac{1}{r} \sum_{j=1}^{r} p(x|\theta_j^{(i)}). \] (3.4)

### 3.2 Inference for Attribute Data Reliability Growth Model.

We next consider inference for the attribute reliability growth model of Section 2.1. After \(i\) stages of testing we have observed \(D^{(i)}\) and want to obtain the posterior distribution \(\Pi(R_j | D^{(i)})\). As pointed out in Section 3.1, the implementation of Gibbs sampler requires the full set of univariate conditionals \(\Pi(R_j | R_j^{(-j)}, D^{(i)})\), \(j = 1, \ldots, m + 1\). We can write

\[ \Pi(R_j | R_j^{(-j)}, D^{(i)}) \propto \mathcal{L}(R_j; R_j^{(-j)}, D^{(i)}) \Pi(R_j | R_j^{(-j)}, D^{(0)}), \] (3.5)

where \(\Pi(R_j | R_j^{(-j)}, D^{(0)})\) is the conditional prior density of \(R_j\) given by (2.6) and for stages \(j = 1, \ldots, i\)

\[ \mathcal{L}(R_j; R_j^{(-j)}, D^{(i)}) = \mathcal{L}(R_j; R_{j-1}, R_{j+1}, n_j) \propto (1 - R_j)R_j^{n_j - 1}, \quad R_{j-1} < R_j < R_{j+1}. \] (3.6)

It is important to note that the conditional likelihood, \(\mathcal{L}(R_j; R_j^{(-j)}, D^{(i)})\), is constrained over \((R_{j-1}, R_{j+1})\). The full conditional density of \(R_j\) given by (3.5) is not of a familiar form. Thus, to able to draw samples from (3.5), to facilitate the implementation of Gibbs sampler, we need to use some random variable generation method. One such method is the standard rejection sampling method which can be easily implemented in our case.

Following Smith and Gelfand (1992), our rejection method for sampling from the conditional posterior \(\Pi(R_j | R_j^{(-j)}, D^{(i)})\), \(j = 1, \ldots, i\), goes as follows:

**Step 1:** Generate \(R_j\) from the conditional prior density

\[ \Pi(R_j | R_j^{(-j)}, D^{(0)}) = \Pi( R_j | R_{j-1}, R_{j+1}, D^{(0)}) \] given by (2.6).

**Step 2:** Generate independently a 0-1 uniform random variate \(u\).

**Step 3:** Compute the ratio

\[ \frac{\mathcal{L}(R_j; R_{j-1}, R_{j+1}, n_j)}{\mathcal{L}(\hat{R}_j; R_{j-1}, R_{j+1}, n_j)} \] (3.7)

where \(\hat{R}_j\) is the maxima of the conditional likelihood.

**Step 4:** Accept \(R_j\) if

\[ u \leq \frac{\mathcal{L}(R_j; R_{j-1}, R_{j+1}, n_j)}{\mathcal{L}(\hat{R}_j; R_{j-1}, R_{j+1}, n_j)} \]

otherwise reject \(R_j\) and go to step 1 and repeat the process.

We note that each accepted \(R_j\) from the above algorithm, constitutes a sample point from the conditional posterior \(\Pi(R_j | R_j^{(-j)}, D^{(i)})\). The likelihood ratio (3.7) is the acceptance probability implying that those values of \(R_j\)'s drawn from the prior will be more likely to be included in the posterior sample if their likelihood contribution is high. The above rejection
method uses the conditional prior as the importance function [see for example, Mueller's (1992)]. Since the conditional prior densities are truncated beta distributions, we can easily draw from the prior in our case. Also, for \( n_j > 1 \), the maxima of the conditional likelihood is available analytically as

\[
\hat{R}_j = \begin{cases} 
R_{j-1}, & \text{if } \frac{n_{j-1}}{n_j} \leq R_{j-1} \\
\frac{n_{j-1}}{n_j}, & \text{if } R_{j-1} \leq \frac{n_{j-1}}{n_j} < R_{j+1} \\
R_{j+1}, & \text{if } R_{j+1} \leq \frac{n_{j-1}}{n_j}
\end{cases}
\]  

(3.8)

For the case \( n_j = 1 \), \( \hat{R}_j \) can be replaced by the value of \( R_j \) generated at the previous iteration of the Gibbs sampler and the acceptance ratio (3.7) is computed accordingly.

A simplification occurs when we draw from the full conditional posteriors of the future reliabilities \( R_j, j = i+1, \ldots, m+1 \). It is easy to see

\[
\Pi(R_j | R_j^{(-j)}, D^{(i)}) = \Pi(R_j | R_{j-1}, R_{j+1}, D^{(0)}), \text{ for } j = i+1, \ldots, m+1,
\]

(3.9)

implying that, given data after \( i \) stages of testing, the conditional posterior distributions of future reliabilities \( R_j, j = i+1, \ldots, m+1 \), are the same as their conditional prior distributions given by (2.6). This simplifies drawing from these distributions, since they are truncated beta densities.

Given \( D^{(i)} \), Gibbs sampler can be implemented in a straightforward manner, as discussed in Section 3.1, by drawing successively from the full conditional posteriors, \( \Pi(R_j | R_j^{(-j)}, D^{(i)}) \), using the proposed rejection sampling algorithm. We note that the above method can be viewed as a special case of the Gibbs-Importance Sampling algorithm of Mueller (1992) with the sampling importance density is chosen as the conditional prior distribution.

Once we obtain a sample \( R_j^1, R_j^2, \ldots, R_j^r \) from the posterior distribution \( \Pi(R_j | D^{(i)}) \), we can obtain all the marginal posterior distributions \( \Pi(R_j | D^{(i)}) \) for \( j = 1, 2, \ldots, m+1 \), using \( R_j^1, R_j^2, \ldots, R_j^r \). Any of the posterior moments for the \( R_j \)'s can be computed from the posterior sample. For example, the posterior mean

\[
E[R_j | D^{(i)}] \approx \frac{1}{r} \sum_{k=1}^{r} R_j^k
\]

(3.10)

The predictive densities for \( N_j, j = i+1, \ldots, m+1 \), can also be computed as

\[
Pr\{N_j = n_j | D^{(i)}\} \approx \frac{1}{r} \sum_{k=1}^{r} Pr\{N_j = n_j | R_j^k\},
\]

(3.11)

where \( Pr\{N_j = n_j | R_j^k\} \) is given by the geometric distribution (2.7).

### 3.3 Inference for Exponential Reliability Growth Model.

Inference for the variable data case follows along the same lines as in the previous section by using the transformation \( R_t = \exp(- \lambda_t) \). The only change is the form of the likelihood function. We can write the conditional likelihood of \( R_t \), for \( j = 1, \ldots, i \)

\[
\mathcal{L}(R_j; R_j^{(-j), D^{(i)}}) = \mathcal{L}(R_j; R_{j-1}, R_{j+1}, x_j, n_j) = \ln(1/R_j)R_j^{x_j+r(n_j-1)},
\]

(3.12)
where \( R_{j-1} < R_j < R_{j+1} \). Again the fully conditional distributions \( \Pi(R_j \mid R_k, D^{(i)}) \), for \( j = 1, \ldots, i \), are not in familiar forms and therefore the rejection algorithm of Section 3.2 is used with acceptance probability

\[
\frac{\mathcal{L}(R_j; R_{j-1}, R_{j+1}, x_j, n_j)}{\mathcal{L}(\hat{R}_j; R_{j-1}, R_{j+1}, x_j, n_j)},
\]

and the maxima of the likelihood is given by \( \hat{R}_j = \exp(-\lambda_j) \) where

\[
\hat{R}_j = \begin{cases} 
  R_{j-1}, & \text{if } e^{-(1/T_j)} \leq R_{j-1} \\
  e^{-(1/T_j)}, & \text{if } R_{j-1} < e^{-(1/T_j)} < R_{j+1} \\
  R_{j+1}, & \text{if } R_{j+1} \leq e^{-(1/T_j)}
\end{cases}
\tag{3.13}
\]

and \( T_j = x_j + \tau(n_j - 1) \) is the total time on test at the \( j \)-th stage of testing. For the full conditional posteriors of the future reliabilities \( R_j, j = i + 1, \ldots, m + 1 \), (3.9) is still applicable and we can draw from the truncated beta densities.

As before, once we form a sample \( R_1, R_2, \ldots, R_r \) from \( \Pi(R \mid D^{(i)}) \), we can obtain all the marginal posterior distributions of \( \lambda_j \)'s by using the one-to-one transform \( R_j = \exp(-\lambda_j) \), for \( j = 1, \ldots, m + 1 \). The predictive distribution for future \( X_j \)'s, \( j = i + 1, \ldots, m + 1 \), is computed via

\[
p(x_j \mid D^{(i)}) \simeq \frac{1}{r} \sum_{k=1}^{r} p(x_j \mid \lambda_j^k),
\tag{3.14}
\]

where \( p(x_j \mid \lambda_j^k) \), is given by the exponential density (2.11). Inference about reliability at any stage \( j \), for given a mission time \( x \), \( x < \tau \), is also straightforward via the distribution of \( R_j^x \) which is obtained using the sample \( R_j^1, R_j^2, \ldots, R_j^r \). Note that we can also make predictions about \( N_j \)'s for \( j = i + 1, \ldots, m + 1 \). We can write

\[
\Pr\left\{ N_j = n_j \mid R_j \right\} = (1 - R_j^x)R_j^{(n_j-1)r}, \quad n_j = 1, 2, \ldots,
\tag{3.15}
\]

and compute the predictive density by using (3.15) for \( \Pr\left\{ N_j = n_j \mid R_j^k \right\} \) in (3.11).

### 4. Extension to Other Models

#### 4.1 Barlow-Scheuer Reliability Growth Model

Our approach can be easily extended to encompass the Barlow-Scheuer attribute data reliability growth model which was considered in Mazzuchi and Soyer (1993). The Barlow-Scheuer model is an extension of the model presented in Section 2.1 which assumes that failures at all stages are assignable cause failures implying that they can be identified and corrected. The Barlow-Scheuer model categorizes the failures at any stage as assignable cause and inherent failures where the inherent failures can only be eliminated with an advancement of technology.

Thus, the authors assume that the probability of assignable cause failures are nonincreasing over the test stages. The test results at each stage is described by a trinomial model with the inherent failure probability \( q \), and assignable cause failure probability \( p_a \).
Mazzuchi and Soyer (1993) presented a Bayesian analysis of this model by using an ordered Dirichlet type prior on $q$ and assignable cause failure probabilities for all the stages, $p_i$, $i = 1, \ldots, m+1$, which preserves the desired ordering $p_1 \geq \cdots \geq p_{m+1}$. By using $R_i = 1 - q - p_i$, we can write

$$\Pi(R_i, q \mid D^{(0)}) \propto \left[ \prod_{j=1}^{m+1} (R_j - R_{j-1})^{\beta\alpha_{j-1}} \right] (1 - q - R_{m+1})^{\beta\alpha_{m+2}-1} q^{\beta\alpha_0-1},$$

(4.1)

defined over $0 \leq R_1 \cdots \leq R_{m+1} \leq 1 - q$, implying that

$$[q \mid D^{(0)}] \sim \text{Beta}\left(\beta\alpha_0, \beta(1 - \alpha_0)\right)$$

(4.2)

and

$$[q \mid R_j, D^{(0)}] = [q \mid R_{m+1}, D^{(0)}] \sim \text{Beta}\left(\beta\alpha_0, \beta\alpha_{m+2}; 0, 1 - R_{m+1}\right).$$

(4.3)

It follows from the above that $\Pi(R_j \mid R_j \begin{pmatrix} \cdots \end{pmatrix}, q, D^{(0)}) = \Pi(R_j \mid R_{j-1}, R_{j+1}, D^{(0)})$ for $j = 1, \ldots, m$, as given by (2.6) and that $\Pi(R_{m+1} \mid R_{m+1}, q, D^{(0)})$ with

$$[R_{m+1} \mid R_m, q, D^{(0)}] \sim \text{Beta}\left(\beta\alpha_{m+1}, \beta\alpha_{m+2}; 0, 1 - q\right).$$

(4.4)

Full distributional results for $p_i$'s and stage reliabilities, $R_i$'s, were obtained in Mazzuchi and Soyer (1993) as mixtures of ordered Dirichlet distributions of the form as in (4.1). As before, the weights in the mixtures are dependent on gamma functions and computational difficulties arise when $m$ is large. However, the Gibbs sampler with the rejection method presented in Section 3.2 can be easily adopted to this case.

Let $A_j$ denote the number of inherent failures at test stage $j$ and after $i$ stages of testing define $D^{(i)} = \{D^{(0)}, a_1, n_1, \ldots, a_i, n_i\}$. To implement the Gibbs sampler we need to draw from the full conditional distributions $\Pi(R_j \mid R_j \begin{pmatrix} \cdots \end{pmatrix}, q, D^{(i)})$, for $j = 1, \ldots, m + 1$, and $\Pi(q \mid R, D^{(i)})$. As in Section 3.2, for stages $j = 1, \ldots, i$, $\Pi(R_j \mid R_j \begin{pmatrix} \cdots \end{pmatrix}, q, D^{(i)})$ is not of a familiar form and therefore we need to use the rejection method of Section 3.2. It can be shown that the conditional likelihoods for $j = 1, \ldots, i$, are given by

$$\mathcal{L}(R_j \mid R_j \begin{pmatrix} \cdots \end{pmatrix}, q, D^{(i)}) = \mathcal{L}(R_j \mid R_{j-1}, R_{j+1}, q, a_j, n_j) \propto (1 - q - R_j) q^{n_j} R_j^{n_j-a_j-1},$$

(4.5)

defined over $R_j-1 < R_j < R_{j+1}$. Given (4.5), we can use the rejection algorithm with acceptance probability

$$\frac{\mathcal{L}(R_j \mid R_{j-1}, R_{j+1}, q, a_j, n_j)}{\mathcal{L}(R_j \mid R_{j-1}, R_{j+1}, q, a_j, n_j)},$$

where, for $(n_j - a_j) > 1$, the maximum of the conditional likelihood is given by
\[
\hat{R}_j = \begin{cases} 
R_{j-1}, & \text{if } \frac{(n_j - a_j - 1)(1-q)}{n_j - a_j} \leq R_{j-1} \\
\frac{(n_j - a_j - 1)(1-q)}{n_j - a_j}, & \text{if } R_{j-1} < \frac{(n_j - a_j - 1)(1-q)}{n_j - a_j} < R_{j+1} \\
R_{j+1}, & \text{if } R_{j+1} \leq \frac{(n_j - a_j - 1)(1-q)}{n_j - a_j} 
\end{cases} 
\] (4.6)

We note that (4.6) implies \( \hat{R}_j < (1-q) \). For the case \((n_j - a_j) = 1\), again \( \hat{R}_j \) can be replaced by the value of \( R_j \) generated at the previous iteration of the Gibbs sampler and the acceptance probability is computed accordingly.

For testing stages \( j = i + 1, \ldots, m \), the conditional distributions are obtained as \( \Pi(R_j | \hat{R}_j^{(-j)}, q, D^{(j)}) = \Pi(R_j | R_{j-1}, R_{j+1}, D^{(0)}) \) which are given as truncated beta densities. For stage \( m+1 \), \( \Pi(R_{m+1} | \hat{R}_j^{(-(m+1))}, q, D^{(j)}) = \Pi(R_{m+1} | R_m, q, D^{(0)}) \) which is again a truncated beta density of the form (4.4).

To obtain the full conditional posterior of \( q \), we write
\[
\Pi(q | \hat{R}_j, D^{(j)}) \propto \left[ \prod_{j=1}^{i} (1 - q - R_j)^{a_j} \right] (1 - q - R_{m+1})^{\beta_{m+1} - 1} q^{\beta_0 - 1},
\] (4.7)
defined over \( 0 \leq R_1 \cdots \leq R_{m+1} \leq 1 - q \), implying that
\[
\Pi(q | \hat{R}_j, D^{(j)}) = \Pi(q | R_1, R_2, \ldots, R_i, R_{m+1}, D^{(j)}).
\] (4.8)

To draw from \( \Pi(q | \hat{R}_j, D^{(j)}) \), we use again the rejection algorithm of Section 3.2. The maximum of the conditional likelihood \( L(q; \hat{R}_j, D^{(j)}) \propto \left[ \prod_{j=1}^{i} (1 - q - R_j)^{a_j} \right] \) is obtained as \( \hat{q} \) such that
\[
\sum_{j=1}^{i} a_j = \sum_{j=1}^{i} \frac{\hat{q}}{1 - \hat{q} - R_j},
\] (4.9)
and \( 0 \leq R_1 \cdots \leq R_i \leq R_{m+1} \leq 1 - \hat{q} \). We note that the value of \( q \) solving (4.9) must be smaller than \( (1 - R_{m+1}) \).

### 4.2 Weibull Reliability Growth Model.

We next consider the case where \( X_i \), the life-length of the system during the \( i \)-th stage of testing is described by the Weibull failure model
\[
p(x_i | \lambda_i, \phi) = \lambda_i \phi x_i^{\phi - 1} \exp(-\lambda_i x_i^{\phi}),
\] (4.10)
where \( \phi > 0 \) and values of \( \phi > 1 \) imply that \((\lambda_i \phi x_i^{\phi - 1})\), the failure rate during the \( i \)-th stage of testing, is increasing. We will assume that the testing scenario is the same as in Section 2.2. To reflect reliability growth we define, as in Section 2.2, \( R_i = \exp(-\lambda_i) \) and assume that the \( R_i \)'s have the ordered Dirichlet prior (2.2), defined over the simplex \( \{ R_i \ | \ 0 \leq R_1 \cdots \leq R_{m+1} \leq 1 \} \), implying that \( \{ \lambda_i \} \) and therefore, the failure rate, is decreasing over the testing stages.

The likelihood of \( \lambda_i \) and \( \phi \), given \((n_j - 1) \) survivals and a failure, at the \( j \)-th testing stage is
\[
L(\lambda_j; x_j, n_j) = \lambda_j \phi x_j^{\phi - 1} \exp(-\lambda_j x_j^{\phi}) \exp[-\lambda_j \phi (n_j - 1)].
\] (4.11)
If we define \( D^{(i)} = \{ D^{(0)}, x_1, n_1, x_2, x_2, \ldots, x_i, n_i \} \), and use the identity \( \lambda_j = \ln(1/R_j) \), the posterior distribution of \( R_j \) and \( \phi \) can be obtained as

\[
\Pi(R_j, \phi | D^{(i)}) \propto \left( \prod_{j=1}^{m+2} (R_j - R_{j-1})^{\beta_{eq}-1} \right) \left( \prod_{j=1}^{i} \ln(1/R_j) \phi x_j^{\phi-1} R_j^{x_j+(n_j-1)\phi} \right) \Pi(\phi | D^{(0)}),
\]

(4.12)

where \( \Pi(\phi | D^{(0)}) \) is the prior of \( \phi \) which is independent of the prior of \( R_j \). The posterior distributions of \( R_j \) and \( \phi \) can not be obtained in analytical form for any choice of \( \Pi(\phi | D^{(0)}) \), but the MCMC methods presented before can be easily adopted to this case.

In implementing the Gibbs sampler, we will draw from the full conditional distributions \( \Pi(R_j | R_j \neq i, \phi, D^{(i)}) \), for \( j = 1, \ldots, m + 1 \), and \( \Pi(\phi | R_j, D^{(i)}) \). As before, for stages \( j = 1, \ldots, i \), these conditional distributions are not of any familiar forms and therefore we will subscribe to the rejection method of Section 3.2. It can be shown that the conditional likelihoods for \( j = 1, \ldots, i \), are given by

\[
\mathcal{L}(R_j; R_{j-1}, R_{j+1}, \phi, x_j, n_j) \propto \ln(1/R_j) R_j^{x_j+(n_j-1)\phi},
\]

(4.13)

which is defined over \( R_{j-1} < R_j < R_{j+1} \). We can use the rejection algorithm with acceptance probability

\[
\frac{\mathcal{L}(R_j; R_{j-1}, R_{j+1}, \phi, x_j, n_j)}{\mathcal{L}(R_j; R_{j-1}, R_{j+1}, \phi, x_j, n_j)},
\]

where the maximum of the conditional likelihood is given by

\[
\hat{R}_j = \begin{cases} 
R_{j-1}, \text{ if } e^{-1/T_j(\phi)} \leq R_{j-1} \\
e^{-1/T_j(\phi)}, \text{ if } R_{j-1} < e^{-1/T_j(\phi)} < R_{j+1} \\
R_{j+1}, \text{ if } R_{j+1} \leq e^{-1/T_j(\phi)}
\end{cases}
\]

(4.14)

and \( T_j(\phi) = x_j^{\phi}+(n_j-1)\phi \). For stages \( j = i + 1, \ldots, m + 1 \), the full conditionals are given by truncated beta densities, since \( \Pi(R_j | R_j \neq i, \phi, D^{(i)}) = \Pi(R_j | R_j \neq i, R_{j+1}, D^{(0)}) \).

The full conditional posterior distribution of \( \phi \) can be written as

\[
\Pi(\phi | R_j, D^{(i)}) \propto \left( \prod_{j=1}^{i} \phi x_j^{\phi-1} R_j^{x_j+(n_j-1)\phi} \right) \Pi(\phi | D^{(0)}),
\]

(4.15)

implying that \( \Pi(\phi | R_j, D^{(i)}) = \Pi(\phi | R_1, R_2, \ldots, R_i, D^{(i)}) \). Again to draw from \( \Pi(\phi | R_j, D^{(i)}) \), we use the rejection algorithm of Section 3.2. The maximum of the conditional likelihood

\[
\mathcal{L}(\phi, R_j; D^{(i)}) \propto \left( \prod_{j=1}^{i} \phi x_j^{\phi-1} R_j^{x_j+(n_j-1)\phi} \right)
\]

is obtained numerically, but this is a straightforward task, since \( \mathcal{L}(\phi, R_j; D^{(i)}) \) is concave in \( \phi \).
5. ILLUSTRATIONS

Example 1: Attribute Reliability Growth Data.

We consider the attribute reliability growth data used by van Dorp, Mazzuchi and Soyer (1994). The authors assumed the reliability growth model of Section 2.1 and used this data for developing optimal stopping rules during a product development test. The inferences for stage reliabilities and the number of items to be tested in future stages were obtained using the closed form expressions involving ratios of gamma functions. In what follows, we develop inferences using the same data via Monte Carlo methods of Section 3. We will use the same prior information used by van Dorp, Mazzuchi and Soyer (1994) and compare our results with their exact results.

The authors considered a testing scenario of $m = 10$ stages and determined whether the testing should be terminated after each stage. Using cost based utility functions, it was shown that the optimal decision was to stop after the 8th testing stage. The resulting data from this process is given in Table 1. The prior parameters were specified as $\alpha = (.36, .34, .102, .0985, .0128, .0127, .0126, .0125, .0124, .0123, .0122, .0120)$ and $\beta = 50$. The resulting expected reliabilities for stages 1 through 11 are given in the first row of Table 2. The second row of Table 2 gives the exact expected reliabilities obtained by the above authors posterior to the data given by Table 1, that is, the inferences after 8 stages of testing. In the third row of the Table 2, we present our inferences using the MCMC methods of Section 3. The results are obtained after 5000 iterations of the Gibbs sampler. We note that our results are exactly the same as the analytical results except in few instances where the differences are most likely due to round-off errors.

<table>
<thead>
<tr>
<th>Testing Stage $j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_j$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Testing Stage $j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[R_j</td>
<td>D^{(0)}]$</td>
<td>0.360</td>
<td>0.700</td>
<td>0.802</td>
<td>0.901</td>
<td>0.913</td>
<td>0.926</td>
<td>0.939</td>
<td>0.951</td>
<td>0.964</td>
<td>0.976</td>
</tr>
<tr>
<td>$E[R_j</td>
<td>D^{(8)}]$-Exact</td>
<td>0.324</td>
<td>0.638</td>
<td>0.737</td>
<td>0.840</td>
<td>0.856</td>
<td>0.873</td>
<td>0.892</td>
<td>0.912</td>
<td>0.935</td>
<td>0.957</td>
</tr>
<tr>
<td>$E[R_j</td>
<td>D^{(8)}]$-MCMC</td>
<td>0.326</td>
<td>0.638</td>
<td>0.737</td>
<td>0.840</td>
<td>0.855</td>
<td>0.873</td>
<td>0.892</td>
<td>0.912</td>
<td>0.934</td>
<td>0.957</td>
</tr>
</tbody>
</table>

In Figure 1, we present the marginal posterior distributions of the current and future stage reliabilities obtained by the MCMC methods after 8 stages of testing. We note that these distributions are based on 5000 iterations of the Gibbs sampler. The convergence of the Gibbs sampler was determined by using different starting values for $R_j$'s and by monitoring the ergodic averages of the $R_j$'s. The plot of the Gibbs sequences for $R_9$, $R_9$, $R_{10}$ and $R_{11}$ are given in Figure 2. Also, the predictive distributions, $Pr\{N_j = n | D^{(8)}\}$, for $j = 9$ and 10 are shown in Figure 3. These distributions are obtained by using the Gibbs sequence in (3.11).
Figure 1. Posterior Distributions of R₈, R₉, R₁₀ and R₁₁ after Testing Stage 8.

Figure 2. Gibbs Sequences for R₈, R₉, R₁₀ and R₁₁ after Testing Stage 8.
Example 2: Exponential Reliability Growth Data.

We next consider the tank failure data analyzed by Sen and Bhattacharyya (1992). The authors' analysis of the data favors both reliability growth and exponentiality of the time between failures. In what follows we will consider the data from the first 10 stages of testing and make inferences about the failure behavior during the 11th stage. In so doing, we will use the exponential reliability growth model of Section 2.2. The data is given in Table 3 in terms of number of miles accumulated to failure in each stage.

<table>
<thead>
<tr>
<th>Testing Stage j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_j</td>
<td>1</td>
<td>56</td>
<td>195</td>
<td>58</td>
<td>175</td>
<td>208</td>
<td>27</td>
<td>7</td>
<td>52</td>
<td>249</td>
</tr>
</tbody>
</table>

For illustrative purposes, in our analysis we will specify $E[R_j | D^{(0)}]$ for $j = 1, 2, \ldots, 10$ as given in the first row of Table 4 and choose a lower value of $\beta = 15$ in our analysis. The posterior means of reliabilities after 10 stages of testing were obtained using the MCMC approach of Section 3.3 and these are given in the second row of Table 4. These results are based on 5000 iterations of the Gibbs sampler.

Based on the data from 10 stages of testing, predictions can be made about the failure characteristics at stage 11 using the distributions of $\lambda_{11}$ and $X_{11}$ which are given in Figure 4. We
note that the predictive distribution of $\lambda_{11}$ was obtained using the transformation $\lambda_{11} = \ln(1/R_{11})$ and the distribution of $X_{11}$ was obtained using the Gibbs sequence in (3.15).

**TABLE 4**

Prior and Posterior Expected Reliabilities

<table>
<thead>
<tr>
<th>Testing Stage $j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[R_j</td>
<td>D^{(0)}]$</td>
<td>0.300</td>
<td>0.500</td>
<td>0.600</td>
<td>0.650</td>
<td>0.720</td>
<td>0.800</td>
<td>0.830</td>
<td>0.860</td>
<td>0.900</td>
<td>0.920</td>
</tr>
<tr>
<td>$E[R_j</td>
<td>D^{(t)}]$</td>
<td>0.538</td>
<td>0.946</td>
<td>0.976</td>
<td>0.980</td>
<td>0.985</td>
<td>0.989</td>
<td>0.990</td>
<td>0.991</td>
<td>0.993</td>
<td>0.995</td>
</tr>
</tbody>
</table>

![Predictive Distribution of $\lambda_{11}$](image1)

![Predictive Density for $X_{11}$](image2)

*Figure 4. Predictive Distributions of $\lambda_{11}$ and $X_{11}$ after Testing Stage 10.*
REFERENCES


