A FRAMEWORK FOR INCORPORATING STRUCTURAL PRIOR INFORMATION INTO THE RECONSTRUCTION AND RESTORATION OF MEDICAL IMAGES

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Abstract: We propose a Bayesian model for medical image analysis that permits prior structural information to be incorporated into the estimation of image features. Inclusion of prior information is accomplished using the image model described in Johnson (1992). A distinguishing feature of this model is the specification of a hierarchical structure for image generation that explicitly incorporates region parameters. These parameters are the mechanism by which prior information regarding image structures is included into the reconstruction or restoration of the image scene. Importantly, these region identifiers allow prior information to be incorporated in a non-deterministic fashion, thus permitting prior structural information to be modified by image data with minimal introduction of residual artifacts. Furthermore, the resulting statistical model permits formation of previously unidentified structures based on the observed data likelihood.

INTRODUCTION

Likelihood-based methods play an important research role in reconstruction of emission computed tomography (ECT) images, but are only occasionally used in clinical settings. Progress towards their clinical use has been hindered due to the poor performance of the maximum likelihood estimate (MLE), excessive computation times associated with estimation algorithms, and the relatively minor gains in image quality that have been obtained through the use of these methods on real data. Despite these problems, interest in statistical reconstruction algorithms remains strong, largely because of the widely held belief that more accurate modeling of image data should result in more faithful reconstructions.

Many of the problems associated with likelihood-based reconstruction methods derive from the manner in which the image scene is parameterized, and related regularity constraints imposed on the over-parameterized image scene. In typical ECT reconstructions, source distributions are approximated by rectangular grids of pixels, with emission intensities assumed constant within pixels. Low resolution 2-D ECT images may contain $64^2$ pixels, while higher resolution 3-D image scenes may contain $128^3$ or $256^3$ pixels. In any case, the number of pixels in the image array is typically of the same order of magnitude as the number of available observations, and so asymptotic properties often attributable to statistical estimates like the MLE do not pertain. In short, the specified parameter space alone is incongruous with the number and quality of data obtained.

To overcome this problem of excessive dimensionality, a number of approaches have been taken to constrain the parameter space. In their pioneering work on the specification of a Poisson model for positron emission tomography (PET), Shepp and Vardi (1982, see also Vardi et al. 1986, Lange and Carson 1984) proposed an estimation-maximization (EM) algorithm to obtain the MLE of the image scene. To impose regularity on this solution, they initialized the image scene with a uniform value, and then stopped EM iterations before the algorithm converged to the MLE. Thus, the uniform initialization was used to implicitly smooth the image scene.

Subsequent researchers have tested more sophisticated approaches to imposing regularity constraints on the image scene (Snyder et al. 1985), with many utilizing the theoretical framework provided by the Bayesian paradigm (e.g. Geman and Geman 1984, Geman and McClure 1987, Hebert and Leahy 1989, Johnson et al. 1990, Johnson et al. 1991a). However, results using these techniques have been mixed. Generally speaking, these approaches sacrifice local image contrast to improve noise properties. Within the Bayesian context, this trade-off arises from a desire to specify prior densities in a mathematically convenient form, rather than specifying priors that reflect likely structures present in the image scene. Thus, smoothing constraints operate equally on all parts of the image scene, and expected areas of contrast are often not distinguished from background noise.

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The purpose of this paper is to detail preliminary experiments with a class of prior models aimed at resolving this paradox. In particular, a general framework is described for specifying prior densities on object distributions using a priori structural information regarding likely features of the image scene. The anticipated sources of such prior information include standard anatomical atlases and, less frequently, cross-correlated images of the same object obtained from other image modalities.

Before describing this class of prior densities, we first briefly review other work in this direction. Following this discussion, we describe a class of prior models specified on all possible partitions of the image scene, and illustrate how these models can be straightforwardly modified to incorporate information from an arbitrary template in a flexible and non-binding way. In the penultimate section, experiments illustrating the utility of the methodology are described in the image restoration setting, and samples obtained from prior densities without image data are used to demonstrate the nature of the constraints imposed by the priors. Finally, a discussion of results appears in the final section.

2. PREVIOUS TECHNIQUES FOR INCORPORATING PRIOR STRUCTURAL KNOWLEDGE

Within the Bayesian paradigm, two important approaches for incorporating specific a priori image information into image restorations/reconstructions have been proposed in recent years. One involves the use of deformable templates as described in, for example, Amit et al. 1991, Chow et al. 1988, and Grenander and Kenan 1986. In the other approach, prior densities of line (or edge) sites incorporated into Gibbs priors are modified to enhance the probability of edges at or near known locations of edges within the object scene (e.g. Johnson et al. 1991b, Leahy and Yan 1991, Gindi et al. 1991). Outlines of these techniques follow.

2a. Deformable templates

The key assumption behind deformable template models is that the true object distribution can be well approximated by a "smooth" deformation of a known template. To illustrate, consider Figure 1 where a template of the letter C is shown in dashed lines, and an observation of the letter C is depicted in bold. Under the assumptions of the model proposed by Amit et al. 1991, the displacements of nearby points between the template and true scene are assumed to be governed by a Gaussian process. For example, consider the displacements of points x and y, denoted by $d_x$ and $d_y$, between the template and observed scene. According to the Gaussian model for these displacements, the difference $|d_x - d_y|$ is assumed to follow a mean zero Gaussian distribution whose variance $\sigma^2_{xy}$ decreases to zero as $x$ approaches $y$. Furthermore, similar Gaussian distributions are assumed to apply simultaneously throughout the image to every pair of points (not only those along the letter C).

![Figure 1](image1.png)

**Figure 1.** Illustrations in deformations in template of letter C.

![Figure 2](image2.png)

**Figure 2.** Configuration of line sites that leads to loss of contrast. Smoothing can occur through broken border, which is difficult to prevent without global neighborhood structure.

The posterior distribution of the image scene can be sampled in two-dimensional problems as the solution to a stochastic differential equation (SDE) (Amit et al. 1991). Likely samples are those that smoothly deform
the template to the observed scene, minimizing both (i) differences in nearby displacements of points between the template and the true scene, and (ii) differences in the displaced template intensities and the observed image intensities. Posterior estimates of restored image scenes represent averages of displaced template intensities and observed pixel intensities.

An important property of deformable template models is that they provide for direct structural interpretation of the image scene. In Chow, Grenander, and Keman’s (1988) study of human hands, finger tips are mapped to finger tips, palms to palms, and so on. Thus, segmentation of the image scene and related image tasks can be easily accomplished by simply examining the map from the template to the estimate of the image scene. This property represents a major advance towards automated image analysis, at least for image scenes amenable to this type of modeling.

Although much effort is currently being devoted to deformable template models, a number of problems concerning their implementation remain unresolved. Presently, these models have not been successfully extended to three-dimensional image scenes, due primarily to difficulties encountered in determining the SDE’s required to obtain posterior samples from the image scene. Also, the model framework discourages sharp local discrepancies of the template from the true scene. Thus local detail in image structure can be lost. However, the most serious deficiency of deformable template models for use in medical image analysis is the property that abnormalities not represented in the prior template are lost or degraded in the posterior estimate of the image scene. Hence, regions of primary interest may be disguised rather than enhanced using this technology. Nonetheless, deformable template models offer great potential for many imaging applications.

2b. Gibbs priors utilizing line sites

A basic difficulty encountered when a Gibbs model is employed for an image scene is that nearby pixel intensities tend to be smoothed, regardless of whether or not they belong to a common anatomical region. Two approaches have been taken towards resolving this difficulty. In the first, the potential functions defining the Gibbs priors are assumed to be bounded, thus limiting the amount of smoothing that occurs between highly disparate pixel intensities (Geman and McClure 1987, Hebert and Leahy 1989). The other approach requires that auxiliary variables called line sites be introduced between each pair of neighboring intensity sites. When these sites are “on”, no smoothing is imposed between the pixels separated by the activated line site (e.g. Geman and Geman 1984, Johnson 1991a). The values of line sites are estimated from the data. It is the latter framework that proves most useful for incorporating a priori structural information.

Given exact knowledge of boundaries of regions in a degraded image, it is trivial to incorporate such knowledge into image restoration/reconstruction using a Gibbs model with line sites (GMLS). All that needs be done is to activate all line sites separating pixels across a known boundary. Smoothing then occurs only within homogeneous regions, and the image can in principle be smoothed to an arbitrarily high degree within regions without loss of contrast between regions.

The difficulty with this method is that the exact locations of boundaries within an image scene are seldom known in advance. Possible sources of specific a priori information in ECT reconstructions are anatomical atlases that can serve as templates, and cross-correlated images obtained from high resolution structural images like MR or CT. In both cases, it is unrealistic to assume that the prior boundaries will exactly correspond to true boundaries in the source distribution.

Incorporating imprecise or approximate boundary information into GMLS is troublesome, and there are a several options available in attempting to deal with prior boundary imprecision. If uncertainty in the boundary location is ignored, then the prior will either cause an artificial boundary in the image – resulting in shifted boundaries or edge artifacts, or it will have no effect – leading to possible loss of contrast due to smoothing in low contrast regions.

Alternatively, the potential functions in some neighborhood along the prior boundary might be modified. However, if all line sites are encouraged to form independently of the values of the others, ringing artifacts can result. Thus, potential functions along the boundary must somehow discourage too many line sites from activating, but at the same time must insure that the activated line sites connect with one another to form closed regions (in order to avoid smoothing “leaks”; see Figure 2). This conflict makes specification of prior
boundary potential functions impossible in small cliques and unmanageable in larger ones. In particular, requiring that (shifted) boundaries form closed regions seems to require a global neighborhood structure.

3. INCORPORATION OF PRIOR STRUCTURAL INFORMATION USING PARTITIONING SETS

The statistical model that I propose for incorporating prior structural information into image restoration/reconstruction is based on the statistical model described in Johnson (1992). The essential features of this model are that the image scene is supposed to be comprised of a number of distinct objects or regions, and is parameterized accordingly. Thus, the posterior distributions of quantities like region volumes and region means are available directly from posterior samples of the image scene. Segmentation of the image into connected regions occurs automatically within the estimation algorithm.

Prior structural information is imposed by encouraging regions to form within the dilation sets of regions obtained from the segmented prior template. Details of how this is accomplished are described in Section 3.2, which follows the description of the basic model formulation in Section 3.1.

3.1 Basic Model Formulation

The statistical model for image generation is specified hierarchically within the Bayesian paradigm. At the lowest level in the hierarchy, a probability distribution on the space of all possible partitions of the discretized image scene is specified within the context of a Gibbs distribution. In the second level of the model hierarchy, random variables representing emission intensity are associated with each partitioning element or region. Individual pixel intensities are then assumed to be drawn from a gamma distribution with mean equal to the region mean in the third stage, and in the final stage of the model projection data are assumed to be generated from Poisson distributions with means equal to weighted sums of pixel intensities.

Denote the array of pixels by $\Xi = \{\xi_i\}$, and assume that the true scene is comprised of an unknown number of intensity-differentiated objects. With each possible configuration of objects associate a partition of $\Xi$, where we define a partition as any collection of sets of connected pixels in which each pixel appears in one and only one set. A set of pixels will be considered connected if it is possible to move from any pixel in the set to any other pixel in the set without leaving the set, where movement between pixels that touch at a corner, edge, or face is permitted.

To facilitate the definition of a probability distribution on the class of all partitions, assign to every pixel in the array $\Xi$ an integer such that all pixels in each partitioning set are assigned the same integer, and each partitioning set is associated with a distinct integer. The particular integers chosen are otherwise arbitrary. These integers are called region identifiers and the array of region identifiers is $R = \{r_i\}$.

A Gibbs distribution can be defined on the array of region identifiers by specifying a neighborhood system, cliques, and potentials. A neighborhood system on an array $\Xi$ is defined to be any collection of subsets $G = \{\xi, \xi \in \Xi\}$ such that $\xi \notin G_\xi$, and $\xi_1 \in G_\xi_2$ if and only if $\xi_2 \in G_\xi_1$. A clique is defined as any subset of $\Xi$ in which every element is a neighborhood of every other element. Denote the set of cliques by $C$. (For further details on the use of Gibbs distributions as prior models for image scenes, see for example, Geman and Geman 1984, Besag 1986, or Johnson et al 1991a.)

The array of pixels may be defined on an arbitrary lattice, although in most imaging models pixels are assumed to lie on a rectangular lattice. However, hexagonal arrays of pixels are considered here, due primarily to their more symmetric neighborhood structure and reduced directional artifacts. A first-order neighborhood set on a hexagonal array is depicted in Figure 3. The subarray in Figure 3a shows the six first order neighbors of the central pixel. Up to arbitrary rotations, the two clique types that result from this neighborhood structure are depicted in Figure 3b.

Given the array $\Xi$ and a neighborhood system $G$, a Gibbs distribution on the region identifier array $R$ must have a density function expressible in the form

$$\pi(R) = \frac{1}{Z} \exp\{-U(R)\}, \quad U(R) = \sum_{c \in C} V_c(R).$$

In this expression, $Z$ is called the partition function and is a constant independent of $R$. $U(R)$ is called the
Figure 3. a) First order neighborhood in a 2D hexagonal lattice. b) Cliques resulting from the first order structure depicted in (a).

energy function, and the functions $V_C$ are called potentials. Importantly, the potential $V_C$ can depend only on components of $R$ that belong to the subscripted clique $C$. An important consequence of this form of the energy function is that the conditional distribution of any component of the system, say $r_i$, given the values of all other components of $R$, depends only on changes in potential functions associated with cliques that contain site $r_i$.

To specify the prior density on the class of partitions, it therefore suffices to specify a neighborhood system (and by implication an associated clique system) and potential functions. For our purposes, the neighborhood system is defined as the entire graph, so that every region identifier is in the neighborhood of every other region identifier. This neighborhood system would generally make the implied distribution computationally intractable since the conditional distribution of any site could well depend on every other site. However, the nonzero potential functions employed in this model are easily computed and the large neighborhood system in fact poses little computational difficulty.

The Gibbs distribution on the class of image partitions can be used to model prior notions regarding likely configurations of objects in the true scene. Three such properties are modeled here. First, configurations having large numbers of regions are discouraged. In noisy images, this type of constraint is needed to prevent individual pixels from assuming completely arbitrary values, a problem that makes maximum likelihood estimation unattractive in the emission computed tomography setting (e.g. Vardi et al, 1985). Second, irregular object shapes are discouraged, although the extent to which they are penalized depends on the nature of the true scene. Third, configurations in which objects are not connected are prohibited.

The first potential type, designed to restrict the number of regions, is a function of the entire graph. Based on previous investigations, we take the form of this potential to be $\alpha K$, where $K$ is the number of regions or partitioning sets in the given configuration and $\alpha$ is an arbitrary hyperparameter. Note that the effect of this potential is to impose a constant penalty of $\exp(\alpha)$ on the formation of each new region.

The second type of potential is employed to impose regularity on the shapes of regions. This potential function imposes a penalty of $\phi$ on each second-order clique when the exterior six pixels are not connected within the outer ring, by region. For example, the configuration depicted in Figure 4a has zero energy according to this potential, while the configuration in Figure 4b is assigned energy $\phi$. Note that there are numerous configurations having both large and small regions that are not penalized under this form of potential.

The third constraint is implicitly enforced in the updating scheme by not updating region identifiers of pixels when such a change would disconnect pixels assigned to the given region.

Given a configuration of region identifiers, or equivalently a partition of the image into objects, the next
Figure 4. The configuration depicted in (a) does not incur a regularization penalty since the pixels labeled 2 are connected even when the center pixel is removed from the graph. In contrast, the configuration depicted in (b) results in a penalty of $\phi$ since the pixels labeled 2 would no longer be connected if the center pixel was removed. In both illustrations, all pixels not outlined are assumed to be assigned to the region associated with identifier 1.

Stage in the model associates with each partitioning set an intensity parameter. Individual pixel intensities within regions are assumed to be drawn from a gamma distribution centered around this value. Gamma distributions are employed because they are the conjugate family for the Poisson distribution (used in the data model below) and because they impose positivity constraints on pixel emission intensities.

In the final stage of the model, counts are assumed to be generated from individual pixels according to Poisson distributions. We denote the probability that a count emitted from a pixel $i$ is registered at detector $j$ by $p_{ij}$. We also assume that the registration probabilities of distinct photons are independent and that the probabilities are known a priori.

Specifically, let $\mu_k$ denote the mean intensity of pixels in partitioning set $k$, let $\Lambda = \{\lambda_i\}$ denote the array of pixel intensities, and let $Y = \{y_j\}$ denote the array of observed Poisson counts at the detector. Then given the region identifiers, the hierarchical model for image generation can be expressed

$$y_j \mid \lambda \sim \text{Poisson}(\sum_i p_{ij} \lambda_i),$$

$$\lambda_i \mid r_i = k, \mu_k, \nu \sim G(\mu_k, \nu).$$

In this parameterization, $G(\mu, \nu)$ denotes a random variable with density function

$$g(\lambda; \mu, \nu) = \frac{1}{\Gamma(\nu)\mu^\nu} \left(\frac{\nu\lambda}{\mu}\right)^\nu \exp\left(-\nu\lambda/\mu\right).$$

Given an observation vector $y$, the model formulation permits the full conditional distributions to be derived in a straightforward manner. For the intensity parameters, these conditional distributions may be expressed

$$\lambda_i \mid y_i, r_i, \mu_r, \nu \sim G\left(\frac{(\mu_r/\nu) y_i + \mu_r}{\mu_r/\nu + 1}, y_i + \nu\right),$$

$$p(\mu_k \mid \Lambda, R) \propto \exp\left[-\frac{\sum_{r_i=k} \nu \lambda_i}{\mu_k} - \sum_{r_i=k} \nu \log(\mu_k)\right],$$

$$p(\nu \mid \Lambda, R, \mu) \propto \pi(\nu) \prod_i \frac{(\nu \lambda_i/\mu_r)^\nu}{\Gamma(\nu)} \exp\left(-\frac{\nu \lambda_i}{\mu_r}\right).$$
The $p_d$ terms have been set to one in the above expressions for ease of exposition. Note that the conditional distribution of the components of $\mu$ follow inverse gamma distributions, but the conditional density of $\nu$ is not a standard form. However due to the large number of terms contributing to the distribution of $\nu$, its distribution tends to be approximately normal and so can be sampled using rejection sampling methods with Student $t$ densities. Note that a vague prior is assumed for $\mu$, and that the prior density for $\nu$, denoted $\pi(\nu)$, is left unspecified.

The conditional distribution of $R_s$ is similarly straightforward and follows a Gibbs distribution. In the Poisson-gamma model, the conditional distribution of the unlabeled pixel in Figure 5, say pixel $\xi_i$, follows a multinomial distribution with probabilities proportional to

$$
\Pr(r_i = 1) \propto \exp\left(-\nu \lambda_1/\mu_1 - \nu \log(\mu_1)\right), \\
\Pr(r_i = 2) \propto \exp\left(-\nu \lambda_1/\mu_2 - \nu \log(\mu_2) - \phi\right), \\
\Pr(r_i = 3) \propto \exp\left(-\nu \lambda_1/\mu_3 - \nu \log(\mu_3) - 5\phi\right).
$$

Figure 5. Hypothetical configuration of region identifiers.

3.2 Inclusion of Prior Region Information

There are at least two sources of specific prior segmentation information that can be routinely obtained for ECT images. The first source arises from standard anatomical atlases. These atlases can be translated and scaled to match general patient dimensions, or alternatively can be matched using more sophisticated methods, for example deformable templates as described by Amit et al. (1991). Note, however, that templates by themselves are seldom sufficient for ECT reconstructions since interest generally lies in the detection of anomalies that by definition are not represented in standard templates.

The second source of prior segmentation information consists of anatomical region information obtained from high resolution, cross-correlated (i.e. superimposed) magnetic resonance (MR) or X-ray computed tomography (CT) images. Although MR and CT images provide anatomic rather than metabolic maps, it is often the case that anatomic and metabolic regions coincide. Because such cross-correlated images are patient specific, they potentially provide a more accurate source of prior information. On the other hand, their use requires an additional imaging study and specialized software to match the slices from the two image modalities. Additionally, the high resolution background image must be segmented, which is often a non-trivial task.

In either case, assume that a background image or template is available, and that this prior image has been scaled, translated, and rotated to approximately match the observed image data. Also, assume that
the background image has been segmented into \( k \) partitioning sets, denoted \( S_1, S_2, \ldots, S_k \). Let the number of pixels in each of these sets be denoted \( c_1, c_2, \ldots, c_k \).

To incorporate information from the segmented background image into the prior density described in Section 3.1, a pseudo-potential value is defined for every pixel and prior region combination. For the \( i^{th} \) image pixel and \( j^{th} \) prior partitioning set, denote this pseudo-potential value by \( v_{ij} \). Because positive potentials in Gibbs distributions correspond to high energy states (low probabilities), pseudo-potentials of pixels inside prior partitioning sets may be assigned negative values, while positive values may be assigned to pseudo-potentials of pixels lying outside of the region.

The purpose of pseudo-potentials is to encourage the formation of regions similar in shape to the regions present in the background image. A naive way of accomplishing this is to assign a potential of

\[
\sum_{i \in T} \sum_{j=1}^{k} \min \left[ \sum_{i \in T} v_{ij}, 0 \right]
\]

(3.2.1)

to each configuration of the image scene. Here, the first sum extends over all partitioning sets \( T \) in the given configuration.

One problem with this form of pseudo-potential and potential assignment is that prior partitioning sets may not align perfectly with regions in the observed data, for reasons stated above. Thus, it is important to allow for shifts in the locations of prior partitioning sets and deformations of their shapes. As an extreme example of this, suppose that an anatomical region two pixels in area is known to exist in the observed image, but that the template has been mis-registered by three pixels. If the negative pseudo-potentials associated with this prior region are assigned only to corresponding pixels contained in that set, then the mis-registration would cause prior knowledge of that region to be lost, since none of the pseudo-potentials associated with the prior region would favor its formation.

To account for such effects, negative pseudo-potentials associated with a prior partitioning set can be extended to pixels outside of that set. For example, negative pseudo-potential values for a prior region can be assigned to all pixels in a fixed dilation radius around the region. Here, these pseudo-potentials increase linearly as the minimum distance from the prior region increase, until a given positive threshold is reached.

Another problem with the potential form (3.2.1) is that small subregions within larger prior partitioning sets are encouraged whenever their boundaries do not extend too far out of the given prior region. Since only regions of similar size need be encouraged, (3.2.1) should be modified to account for the relative sizes of the prior and estimated region volumes. If \( \kappa(i) \) denotes the number of pixels in a partitioning set \( i \), such effects can be included in the prior region potential by modifying (3.2.1) to

\[
\sum_{i \in T} \sum_{j=1}^{k} \min \left[ \delta \frac{\kappa(i) - c_j}{c_j} + \sum_{i \in T} v_{ij}, 0 \right],
\]

(3.2.2)

for some positive constant \( \delta \). Again, the set \( T \) represents all partitioning sets in the configuration. The purpose of the first term is to prevent regions having disparate areas or volumes from being assigned increased prior probability. Thus only regions having shapes and areas similar to a prior partitioning set are encouraged. The degree of similarity required for a region to be assigned higher prior probability is determined by the prior parameters \( v_{ij} \) and \( \delta \).

As a simple illustration of the combined effects of all potentials specified in determining the conditional probabilities for region identifier updates, consider a simple scene supposed a priori to be composed of two regions. Assume that the pseudo-potential values for each of the two prior partitioning sets are assigned as depicted in Figures 6a and 6b. Suppose also that the current configuration of the estimated scene is as shown in Figure 6c, and that the region identifier of the shaded pixel is to be updated. Let the two prior partitioning sets be labeled 1 and 2, and the partitioning sets currently in the estimated image be labeled 3 and 4. Note that with the shaded pixel excluded from both estimated regions,

\[
\sum_{i \in 3} v_{i}^1 = -90, \quad \sum_{i \in 4} v_{i}^1 = 3, \quad \sum_{i \in 3} v_{i}^2 = 57, \quad \sum_{i \in 4} v_{i}^2 = -15,
\]
and

\[ c_1 = 53, \quad c_2 = 12, \quad \mathcal{N}(3) = 52, \quad \text{and} \quad \mathcal{N}(4) = 12. \]

Given this configuration, the shaded pixel can be assigned to one of three partitioning sets - the sets associated with region identifiers 3 and 4, or a new partitioning set containing only the shaded pixel, say region 5. If it is appended to one of the existing regions, there are only two distinct partitioning sets in the resulting configuration, so the global potential for region number is 2. If it forms a new region, the global potential for number of regions is 3. No irregularity penalty is incurred if the pixel is assigned to regions 3 or 5, but a potential of \( \phi \) is incurred if it is assigned to region 4. Finally, the potential function (3.2.2) results
in the following potential values for assignment of the pixel to regions 3, 4, and 5 (assuming that $\delta = 1$).

\[
\text{Potential if } 3 = \sum_{i=3}^{4} \sum_{j=1}^{2} \min \left[ \sum_{k \in i} v_k + \delta \left( \frac{N(t) - c_i}{c_j} \right)^2, 0 \right] \\
= \min \left[ -91 + 0^2, 0 \right] + \min \left[ 57 + \left( \frac{41}{12} \right)^2, 0 \right] + \min \left[ 3 + \left( \frac{41}{53} \right)^2, 0 \right] + \min \left[ -15 + 0^2, 0 \right] \\
= -106
\]

\[
\text{Potential if } 4 = \min \left[ -90 + \left( \frac{1}{52} \right)^2, 0 \right] + \min \left[ 57 + \left( \frac{40}{12} \right)^2, 0 \right] + \min \left[ 3 + \left( \frac{40}{53} \right)^2, 0 \right] + \min \left[ -15 + \left( \frac{1}{12} \right)^2, 0 \right] \\
= -104.99
\]

\[
\text{Potential if } 5 = \min \left[ -90 + \left( \frac{1}{52} \right)^2, 0 \right] + \min \left[ 57 + \left( \frac{40}{12} \right)^2, 0 \right] + \min \left[ 3 + \left( \frac{41}{53} \right)^2, 0 \right] + \min \left[ -15 + 0^2, 0 \right] \\
\quad + \min \left[ -1 + \left( \frac{52}{53} \right)^2, 0 \right] + \min \left[ 0 + \left( \frac{1}{12} \right)^2, 0 \right] \\
= -105.04
\]

Thus, the prior structural information increases the probability of the assignment of the shaded pixel to region 3 by approximately one unit on the log-likelihood scale.

The relative probabilities for assignment of the shaded pixel to the three possible configurations are therefore proportional to

\[
\Pr[3] \propto \exp(-2\alpha + 106), \quad \Pr[4] \propto \exp(-2\alpha - \phi + 104.99), \quad \text{and} \quad \Pr[5] \propto \exp(-3\alpha + 105.04).
\]

Inclusion of the data likelihood is incorporated as in Section 3.1.

4. EXAMPLES

Because of the hierarchical nature of the model formulation, it is possible to sample from the prior density on an image scene before introducing data. Although seldom done in image analysis problems, this
technique can often be quite useful in identifying shortcomings of Bayesian models, since it permits visual assessment of prior information imposed on the image scene.

I have developed general software that quickly produces a prior density for any two-dimensional image scene from a given image template. The software can be used on either hexagonal or square pixel arrays, although attention here is restricted to hexagonal arrays of dimension 128 x 148.

The template used represents a 128 x 128 bitmap of the Hoffman phantom (Hoffman et al 1990) converted to a 128 x 148 hexagonal array. The hexagonal array is depicted in Figure 7a. A Poisson observation of this image is depicted in Figure 7b. In this figure, exterior grey matter pixels have mean 20, interior white matter pixels mean 10, and ventricle regions mean 2. The background intensity is 0. At each pixel, independent Poisson observations were generated with the pixel means illustrated in Figure 7a.

A critical element in setting the prior on the image scene involves choosing values for the hyperparameters \( \alpha, \phi, \psi = \psi^2 \), and delta. Previous experiments involving this model framework for image restoration (Johnson 1992) indicate that values of \( \alpha \) in the range (0.0, 14.0) produce acceptable restorations. Similarly, values of \( \phi \) in the range (1.0, 5.0) tend to work well for this class of images.

In principle, a large number of brain studies using human patients and the Hoffman phantom template could be conducted to estimate "optimal" values of the parameters \( \psi \) and \( \delta \). However, such studies would be extremely time consuming in practice and would require that specific optimality criteria be established. Indeed, the optimal parameter values could well depend on the imaging task being performed. To avoid these difficulties at this preliminary stage of investigation, a more heuristic approach is taken here.

Because a primary goal in developing this model was to incorporate information from cross-correlated MR and CT images into ECT reconstructions, values of the hyperparameters \( \psi \) were chosen with the types of errors typical of image registration in mind. Currently available software used to match these images has proven capable of aligning the images to within three image pixels (on the specified grids). Thus, pixels within the third-order dilation around each prior region were assigned negative pseudo-potentials. For simplicity, the values of all negative pseudo-potentials within this radius were assumed to be the same for the given region. The values of the pseudo-potentials outside of the third-order dilation set were taken to linearly increase to a threshold equal to the absolute value of the center pseudo-potentials at the sixth-order dilation. This positive threshold value was then assigned to each pixel outside of the sixth-order dilation set. The absolute magnitudes of the pseudo-potentials were determined so that the potential function for a region corresponding exactly to a prior region would just offset the penalty for a new region (\( \alpha \)). If this pseudo-potential value was greater than -1 for a (large) region, it was decreased to -1.

The value of \( \delta \) was also taken to be region specific and was simply taken to be equal to the absolute value of the interior pseudo-potential values for each region.

Samples from the prior distribution for the image scene are depicted in Figures 8a and 8b. The sampled values were obtained using the Gibbs sampler described in Johnson (1992), modified to include the conditional distributions obtained with the addition of specific prior information as described at the end of Section 3.2. The hyperparameters were \( \alpha = 10 \) and \( \phi = 3 \); \( \psi \) and \( \delta \) were selected using the criterion just described. The purpose of the prior simulations is to depict the typical appearance and variability expected from images within this class.

The prior model used to generate the random brains depicted in Figure 8 was then used to restore the Poisson observation of the modified Hoffman phantom shown in Figure 7b.

To assess the utility of various forms of prior information, the observed image was restored using three priors. The first prior represented the baseline model described in Johnson (1992) without any template information but including the cliques and potentials described in Section 3.1. The second prior was the prior used to generate the random images in Figure 8, and was based on the region template that generated the observed image. The third prior was similar to the second, except that the template upon which it was based was shifted two pixels vertically and incorporated two dilated white matter regions. The deformed, shifted template is shown in Figure 9. The third prior was introduced with the goal of assessing the utility of incorporating approximate prior region information, and is intended to simulate the effects of mis-registration and differences between structural and anatomical images. Posterior samples for the image scene appear in
Figure 10. Note that no maximization procedures have been applied to these scenes; the depicted images represent only random samples generated from the posterior distributions obtained using each prior.

As expected, the sampled value obtained using the true prior region information (Figure 10a) appears more similar to the phantom data depicted in Figure 7 than that obtained with the baseline model (Figure 10b). The sampled value obtained using the shifted prior also seems to be a better approximation to the image scene, although its advantages are less clear (Figure 10c).

To more quantitatively assess the performance of the model, posterior distributions of region areas and means can be examined. Figure 11 depicts sequences of region means and areas for two regions indicated in the phantom. These sequences were obtained using the Gibbs sampling technique as described in Johnson (1992). This model facilitates obtaining such samples since regions are estimated as a parameter in the model.

Figure 11a. Sequences representing the last 200 values obtained in the Gibbs sampler for the region areas of partitioning sets “A” and “B”.

Figure 11b provides the results for the region labeled “A” in Figure 7. The estimated posterior means and variances of this region area were (36.0,0.81), (35.7,7.30), and (36.4,0.90) for the models using the true, baseline, and shifted and deformed priors, respectively. The actual area was 36 pixels. The corresponding values for the mean emission rate from this region were (19.00,0.57), (19.16,7.0), and (19.10,0.57). The theoretical value was 20, although the sample mean of the observations was 18.44. Note that in the “shifted and deformed” model, this region was only shifted and was not eroded. In this case, the addition of the specific prior information, whether shifted or not, seemed to improve the precision of the estimation of the region area, but had relatively little effect on the region mean since this parameter was well identified by the model even when the region boundaries were mis-registered.
Similar results are depicted in Figure 11b for region labeled “B” in Figure 9. In this case, however, the third prior assumed that this region had been eroded to 89 pixels in area, when in fact the true area is 140. The sample sequence for the three priors indicates that the prior bias is not overcome by data, suggesting that the form of the penalty for region size in (3.2.2) is too severe or that the value for $\delta$ should be reduced. We are currently experimenting with alternative formulations.

Despite this quantitative bias, the appearance of the restored images appears acceptable, and in fact the bias using the deformed template appears less severe than with the baseline. This is due to the formation of multiple partitioning sets at the location of region “B”. These smaller partitioning sets, which augment the primary high intensity area, are not included in the total region area as computed by the algorithm (only partitioning sets which (would) receive a non-zero potential from (3.2.1) for the given prior partitioning set were counted). The estimated posterior means and variances for the true template, baseline, and deformed template prior models were (20.47,0.10), (20.45,1.10), and (19.92,0.73), respectively. The theoretical value was 20, although the sample mean of the observations observed in the region was 20.48.

Finally, the posterior probability for the existence of a region corresponding to a prior partitioning set can be estimated using this model. If the prior partitioning set is again defined to be present in the sampled images when a region in the sampled scene receives a non-zero value according to the potential function (3.2.2), then the probability of the presence of a region, say the region labeled “C” in Figure 7, can be estimated from the sampled sequence of images. In this case, the estimated probabilities for the presence of region “C” based on the last two hundred sampled images in the sequence were 1.0, 0.76, and 1.0 for the true template, baseline, and deformed template prior models. Of course, in more extensive sampling from the posterior distribution the probabilities would not be 1.0 for either the true or deformed template models, and the sampled images cannot be considered independent. Nonetheless, both templates appear useful in detecting small regions that appear in the template, even when they are not present at the exact location specified by the template.

5. DISCUSSION

The model is under development and numerous issues concerning its implementation remain unresolved. For example, the choice of (3.2.2) as the functional form for encouraging the formation of regions in the posterior distribution is highly suspect and was intended only to assess the viability of the model framework. More refined functions for accomplishing this goal are clearly needed, as are better selection criteria for components of $\nu$ and other model hyperparameters. Still, the experiments described in the previous section indicate that this approach offers the potential for incorporating vague prior structural information into image restoration. Additionally, estimates obtained from the image scene are directly interpretable and posterior uncertainty associated with image features can be assessed.

In reconstruction settings, the sampling scheme used to estimate the posterior distributions in the above restoration setting is not computationally feasible. Alternative techniques to approximate the posterior distributions using methods related to iterative conditional averages (ICA, Johnson 1991) are currently under investigation (Jiang et al 1992).

6. ACKNOWLEDGEMENTS

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REFERENCES


Figure captions for photographic plates

Figure 7. a) Hoffman phantom. This image represents a slice from a 128² bitmap of a 3-D brain phantom. The original bitmap was converted to a 128 x 148 hexagonal lattice, and then converted back to a 512 x 512 rectangular lattice for display. Statistical modeling and processing were performed on the hexagonal array. b) A Poisson observation of the Hoffman phantom (a) in which the background intensities has mean rate 0, the interior ventricles mean 2, the grey matter mean 10, and the perimeter white matter mean 20. In this and all other images presented, the slight artifacts near the edges of regions and the image boundary result from the conversion algorithm used to transform between hexagonal and rectangular arrays.
Figure 8. Sampled images from the Gibbs prior for the region identifier matrix. The colors associated with each partitioning set were chosen randomly. The initial value for image Gibbs sampler was the segmented Hoffman phantom shown in (7a). Defining an iteration to be 128 × 148 updates (the number of pixels in the array), the images correspond to the 500th (a) and 1000th iteration of the Gibbs sampler.

Figure 9. The shifted, deformed template used as the third prior model. Note that indicated regions have been eroded (as compared to Figure 7) and that the image has been shifted two pixels vertically. Colors assigned to regions were assigned randomly.

Figure 10. Random posterior samples of the image scene. All images represent the 1000th iteration in the Gibbs sampler. a) The sample obtained using the prior model depicted in Figure 8. a) The sample obtained using the baseline without template information. a) The sample obtained using the shifted and deformed prior depicted in Figure 9.