INSPECTION TIMES FOR STAND-BY UNITS

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SUMMARY

This paper studies optimal timing of inspections for units in cold stand-by. The problem is approached in continuous time. The policies developed adapt optimally to information provided by failures of the main unit and by aging of both the stand-by and main unit.

Results include a general form for the optimal policy, consisting of a recursive relation between successive inspection times, and of a renewal-type integral equation to determine the value of the objective function for a given policy. Further results are derived for the case in which the failure densities describing the process are log-concave. Then the stopping rule is easily characterized. It is also shown that the interval between inspections decrease with time, and that the optimal schedule is unique. A binary search algorithm for finding the solution is outlined.

Key Words: Reliability, Inspection, Constrained Optimization, Nonmarkovian Deterioration.
1. INTRODUCTION

Units in cold stand-by are often installed to improve the reliability of a system, and they are brought into operation only if the normally operating (or main) components fail. Power generators in hospitals and cooling systems in nuclear power plants are typical examples. Units may deteriorate or become inoperative even if not used, and it is typically needed to inspect stand-by units for failures. Clearly, frequent inspections reduce the chance that a stand-by unit will fail to operate when necessary, but are costly. Proper scheduling of inspection times is therefore a very important safety and cost management problem. From a more theoretical point of view, scheduling raises some interesting modelling and optimization questions, some of which are not yet addressed in the literature in due generality.

This paper is concerned with optimal policies for the inspection of units in cold stand-by. The problem is studied in continuous time, in the tradition of Barlow, Hunter and Proschan (1963). The objective is to minimize the number of inspections, subject to a fixed probability of the stand-by not functioning when called into operation. The number of inspections is counted between replacements (or repairs to new) of the stand-by units. The state of the system as a whole is modelled by a semimarkov process, to accommodate for aging of both the main unit and the stand-by unit. The intervals between inspections adapt accordingly. Also, inspection policies account for the information provided by failures of the main unit.

Inspection of stand-by units in continuous time have been studied by Nakagawa (1980), who considered equally spaced inspection times and exponential failures of the main unit. Further models for inspection of stand-by systems have been developed in discrete time. For a review and recent results see Thomas, Jacobs and Gaver (1987). Relative advantages of continuous time modelling as compared to discrete time are discussed in Parmigiani (1993c). In short, discrete time models lend themselves to treatment of multiple stages of deterioration, but not to nonmarkovian aging schemes, that are best handled in continuous time, with enhancement of the applicability as well as of the insight that can be drawn from the form of the optimal solution.
The discussion proceeds as follows. Section 2 introduces the model and the basic terminology. Section 3 gives a general form for the optimal policy, consisting of a recursive relation between successive inspection times, and of a renewal-type integral equation to determine the value of the objective function for a given policy. Section 4 makes the further mild restriction that the failure distributions involved are log-concave, and develops properties of the optimal policy. The single-inspection problem is used as a stepping stone to work out the optimal number of inspections. Then it is shown that the interval between inspections decrease with time, and that the optimal schedule is unique. Also an easy-to-implement binary search algorithm is suggested.

2. Preliminaries

Consider a two-unit system, consisting of one main unit and one unit in stand-by. The system starts at time 0 with both units in functioning conditions. Both units are subject to random failures. Failures of the main unit bring the stand-by unit into operation; for this reason they are called initiating events. Failures of the stand-by unit do not become apparent unless an initiating event occurs, or an inspection takes place. If the stand-by has failed by the time an initiating event occurs, there is a system failure, usually with severe consequences.

The times $X_1, X_2, \ldots$ of the initiating events are generated by a renewal process. In particular, the waiting times between events have support $(0, \infty)$, and are distributed according to a c.d.f. $G$, with continuous density $g$ and failure rate $r_G(y)$. The event times are observed at no cost. The time to failure $Y$ of the stand-by unit has support $(0, \infty)$, c.d.f. $F$, with density $f$, finite expectation $\mu_F$ and failure rate $r_F(y)$. Also, as customary, $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$. Units fail independently of each other. In the remainder, I use the term cycle to indicate the time between two successive failures of the stand-by, and the term subcycle to indicate the time between two successive detected failures — irrespective of the unit failing.

If the stand-by unit is functioning at the time of an initiating event, it operates until the main unit is replaced (or repaired to new). The replacement or repair time is assumed to be negligible to the effects of aging. Also, no failure of the stand by may occur during repair of the main unit. This is a convenient approximation for the common practices of using a further
stand-by, or taking suitable safety measures, during repair. Finally, there are repair costs for the main unit, but these do not depend on the scheduled inspection times for the stand-by unit and can be ignored.

Information on the state of the stand-by unit can be obtained only by a suitable test or inspection. When a failure is detected, the unit is replaced (or repaired to new). The problem is choosing optimally the times for performing the test. The objective is to determine the schedule that minimizes the number of inspections under the constraint of a fixed probability of system failure in a cycle.

The probability of failure within a cycle is the interesting parameter to control is from a reliability point of view. No matter how frequent inspection may be, there exist a nonzero probability of system failure in each cycle. After the stand-by unit is replaced the problem restarts under the same conditions, so that eventually, one such failure will occur with probability one. Note, incidentally, that, under the assumptions of this model, fixing the probability of failure within a cycle is equivalent to fixing the expected number of cycles before a system failure. Also, it is equivalent to fixing the long run expected number of system failures per unit of time.

When an initiating event occurs, it is observed at no cost whether the stand-by is functioning. From the renewal property, knowledge of the time of the latest event (henceforth the $k$-th) is a sufficient summary for the inspection decision. The optimal policy must therefore be defined conditionally on $X_k = x$, and is followed for one subcycle, that is until either a new initiating event occurs —leading to a revised $x$, or the stand-by fails and the failure is detected by inspection before the main unit fails. The adaptive character of the policy differentiates the solution derived here form any of the non-periodic inspection policies consireded in the literature (see Parmigiani 1993a for a review). Some models handling delayed symptoms of failure (Sengupta 1980, Parmigiani 1993b) present some formal similarities with the model studied here. However in such models the initiating event is constrained to occur after time $Y$, and there is no renewal. Also, the objective function is different, the concern being that of avoiding production loss, or detecting a chronic disease as early as possible.
3. Optimal Policies

Consider a single cycle and suppose \( X_k = x \). Let \( U = X_{k+1} - x \); \( U \) has c.d.f. \( G \). Define 
\[
f_x(y) = f(y)/\bar{F}(x), \quad y > x \quad \text{and} \quad F_x(y) = F(y)/\bar{F}(x), \quad y > x.
\]

A policy, or schedule, is any sequence \( \tau = \{\tau_i\}_{i=0,1,2,\ldots} \), where \( \tau_i \) is the time of the \( i \)-th inspection, \( \tau_i > \tau_{i-1} \) and \( \tau_0 = x \); \( n = \sup\{i : \tau_i < \infty\} \) is the number of planned examinations, finite or infinite. If \( n = 0 \), the policy says not to inspect until the end of the subcycle. Denote by \( I \) be the number of inspections actually carried out in a cycle and by \( S \) be the indicator of system failure within the cycle.

The problem is in choosing the sequence \( \tau \) that minimizes the expected inspection cost \( E\{I\} \) subject to \( E\{S\} = p_0 \), where expectations are conditional on all information available at time \( x \). The Lagrangean is \( E\{I\} + \lambda E\{S\} - \lambda p_0 \) where \( I \) and \( S \) depend on the policy adopted. The problem corresponds to the unconstrained optimization of a risk functions in which one pays \( \lambda \) times the cost of one inspection in case of system failure in the cycle. Thus in the context of this problem \( \lambda \) can be taken to be greater than 1 with no loss of generality.

Let the Lagrangean associated with the optimal policy conditional on \( X_k = x \) be denoted by \( \Lambda(x) \). Also, denote by \( \mathcal{L}^T(x) \) the Lagrangean associated with using \( \tau \) in the first subcycle and then continuing optimally in subsequent subcycles until the end of the cycle. The following Lemma gives an expression for \( \mathcal{L}^T(x) \) in terms of the policy adopted.

**Lemma 1**

\[
\mathcal{L}^T(x) = A^T(x) + \int_{0}^{\infty} \int_{0}^{y-x} \Lambda(x + u)g(u)f_x(y)du dy
\]

where

\[
A^T(x) = \sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} \left[ (i+1)\bar{G}(\tau_{i+1} - x) + \sum_{j=0}^{i} j[G(\tau_{j+1} - x) - G(\tau_j - x)] \right] f_x(y)dy \\
+ \sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} \lambda[G(\tau_{i+1} - x) - G(y - x)]f_x(y)dy - \lambda p_0,
\]
Proof: Condition on $X_k = x$ and $Y = y \in (\tau_{i+1}, \tau_i]$. If the event occurs before the failure of the stand-by unit, that is if $X_{k+1} = u < x + y$, we face the same problem, with Lagrangean $\Lambda(x + u)$. Otherwise, we have the following. A system failure occurs if the next event occurs after a failure of the stand-by unit, that is if $X_{k+1} = u + x \in (\tau_{i+1}, \tau_i]$. This occurs with conditional probability $G(\tau_{i+1} - x) - G(y - x)$. On the other hand, the number of inspections performed is $j$ with probability $G(\tau_{j+1} - x) - G(\tau_j - x)$ if $j \leq i$, and $i + 1$ with probability $G(\tau_{i+1} - x)$. Taking the expectation with respect to $Y$ gives (1). $\square$

The following results provide the tools for the optimization. Denote by $\tau(x, \lambda)$ the optimal schedule given $X_k = x$ for a fixed $\lambda$. It follows:

$$\Lambda(x) = \mathcal{L}^{\tau(x, \lambda)}(x)$$

(3)

The evaluation of the Lagrangean is governed by a fundamental integral equation which is analyzed in Lemma 2. If $\lambda = 0$ the Lemma gives the integral equation for the expected number of inspections in a cycle.

Lemma 2 Let $r_F$ be nondecreasing. At the optimum, the Lagrangean is:

$$\Lambda(x) = A^{\tau(x, \lambda)}(x) + \int_0^\infty \Gamma(v, x) A^{\tau(v, \lambda)}(v) dv$$

(4)

where $\Gamma(v, x)$ is the resolvant of

$$\Lambda(x) = A^{\tau(x, \lambda)}(x) + \int_x^\infty \Lambda(v) g(v - x) \tilde{F}_x(v) dv.$$  

(5)

Proof: Replacing (3) into (2) and reversing the order of integration at the right hand side, it follows that at the optimum, the Lagrangean satisfies the integral equation (5). The following is a sketch of the proof of the existence and uniqueness of the solution. Define the nonnegative
kernels

\[ K_n(x, v) = \begin{cases} 
  g(v - x) F_n(v) & \text{if } x < v < n \\
  0 & \text{otherwise}
\end{cases} \]

Consider the equation

\[ \mu \Lambda_n(x) = A^{\tau(x, \lambda)}(x) + \int_0^n \Lambda_n(v) K_n(x, v) dv \tag{6} \]

By a standard inequality (Zabrejko, 1975 p. 76), the spectral radius \( \rho(K_n) \) of the kernel \( K_n \) can be bounded by:

\[ \rho(K_n) \leq \sup_{0 \leq x \leq n} \int_0^n K_n(x, v) dv; \]

in turn

\[ \sup_{0 \leq x \leq n} \int_0^n K_n(x, v) dv < \sup_{0 \leq x \leq n} P(Y > U + x | Y > x) \]

Now since \( r_F \) is increasing, \( Y - x | Y > x \) is stochastically dominated by \( Y \), so that \( P(Y > U + x | Y > x) < P(Y > U) < 1 \). Therefore \( |\mu| > \rho(K_n) \) is satisfied at \( \mu = 1 \) and the solution of (6) at \( \mu = 1 \) is unique. We write it as:

\[ \Lambda_n(x) = A^{\tau(x, \lambda)}(x) + \int_0^n \Gamma_n(x, v) A^{\tau(v, \lambda)}(v) dv \tag{7} \]

where

\[ \Gamma_n(x, v) = \sum_{i=1}^{\infty} K_n^i(x, v) \]

\[ K_n^i(x, v) = \int_x^n \ldots \int_x^n K_n(x, s_1) K_n(s_1, s_2) \ldots K_n(s_{j-1}, v) ds_1 \ldots ds_{j-1} \]

As \( A^{\tau(v, \lambda)}(v) \) is nonnegative and \( K_n^i(x, v) \) are increasing in \( n \), \( \Lambda_n(x) < \Lambda_{n+1}(x) \). Also, from (6), \( \lim_{n \to \infty} \Lambda_n(x) = \Lambda(x) \) if it exists. But for every \( x \), \( \Lambda(x) \leq \lambda[P(S | Y > x) - p_0] \) —the
Lagrangean when no inspections are scheduled. The equality holds when no inspection is the optimal policy. In turn, \( \lambda[P(S|Y > x) - p_0] < \lambda(1 - p_0) \). So \( \Lambda(x) \) is finite for every \( x \), and the limit exists. Therefore, we can take (7) to the limit also and obtain (4). \( \square \)

The optimal schedule for fixed \( \lambda \) obeys a simple recursive form described in Lemma 3.

**Lemma 3** Given \( X_n = x \) and \( \lambda \), the optimal schedule \( \tau \) must satisfy:

\[
\frac{(\lambda - 1)G(\tau_{i+1} - x) - \lambda G(\tau_i - x) + 1}{g(\tau_i - x)} = \frac{\lambda F_x(\tau_i) - (\lambda - 1)F_x(\tau_{i-1}) - 1}{f_x(\tau_i)} \quad i = 1, 2, \ldots \tag{8}
\]

**Proof:** Singling out terms that depend on \( \tau_i \) and rearranging, the Lagrangean in (1) can be written as

\[
\mathcal{L}^T(x) = C + \int_{\tau_{i-1}}^{\tau_i} \left\{ \lambda[H(\tau_{i+1}) - H(y)] + (i + 1)\bar{H}(\tau_{i+1}) + \sum_{j=0}^{i} \lambda[H(\tau_{j+1}) - H(\tau_j)] \right\} f_x(y)dy + \int_{\tau_{i-1}}^{\tau_i} \left\{ \lambda[H(\tau_i) - H(y)] + i\bar{H}(\tau_i) + \sum_{j=0}^{i-1} \lambda[H(\tau_{j+1}) - H(\tau_j)] \right\} f_x(y)dy - H(\tau_i)\bar{F}(\tau_{i+1})
\]

where \( C \) is independent of \( \tau_i \) and, for compactness of notation, \( H(\tau) = G(\tau - x) \). Computing partial derivatives of the above and manipulating gives first order conditions:

\[
\frac{\partial \mathcal{L}^T(x)}{\partial \tau_i} = g(\tau_i - x)\{\lambda[F_x(\tau_i) - F_x(\tau_{i-1}) - \bar{F}_x(\tau_{i-1})] - f_x(\tau_i)\{\lambda[H(\tau_{i+1}) - H(\tau_i)] - \bar{H}(\tau_{i+1})\}
\]

for \( i = 1, 2, \ldots \). Setting to 0 and rearranging gives (8). \( \square \)

In practice, the optimization problem can be solved by iteratively deriving \( \tau(x, \lambda) \) and verifying that under \( \tau(x, \lambda) \) the probability of system failure is \( p_0 \). Equations (8) can be used to determine recursively the optimal solution for any given \( \tau_1 \). Optimization of \( \tau_1 \) must be done numerically, with the help of Lemma 2.
4. Properties of the Optimal Policies

In this section I impose further conditions on the densities $f$ and $g$ and derive properties of the optimal sequence of inspection times. The conditions involved are increasingness of the failure rate and log-concavity: both are quite natural in this context and do not constitute a severe restriction. Log-concavity is stronger than increasingness of the failure rate. The class of log-concave density includes a large number of widely used failure distributions and suitably models wearout (see Barlow and Proschan, 1965).

The first result concerns the so-called single-inspection problem. Fix $\lambda$ and for the moment restrict consideration to strategies with $x_2 = \infty$; the decision space includes now single-inspection policies with inspection time $x_2 = x_1$, as well as no inspection in the subcycle. The following result gives a necessary and sufficient condition for no inspections to be preferred to one inspection. The condition can be verified without having to compute the optimal one-inspection policy. Uniqueness of the inspection time is also guaranteed by the result.

**Theorem 1** Let the failure rates $r_F$ and $r_G$ be nondecreasing. If

\[
\lim_{x \to \infty} \frac{r_G(x)}{f_x(x)} > \frac{\lambda}{\lambda - 1}
\]

it is optimal not to inspect. Otherwise, it is optimal to inspect and there exist a unique optimal inspection time $x^*$, given by the solution of:

\[
g(x - r) [\lambda F_x(r) - 1] = \lambda G(x - r)f_x(r)
\]

**Proof:** The Lagrangean for no inspection in the subcycle is:

\[
L^0(x) = \int_x^{\infty} \left[ \lambda G(y - x) + \int_0^{y-x} \Lambda(u + x)g(u)du \right] f_x(y)dy - \lambda p_0.
\]

The Lagrangean for a one-inspection policy with inspection time $x$ can be computed from (1).
by letting $\tau_2 \to \infty$:

$$
\mathcal{L}^\tau(x) = \int_x^T \left\{ \lambda [\bar{G}(y-x) - \bar{G}(\tau - x)] + \int_0^{y-x} \Lambda(x+u)g(u)du + \bar{G}(\tau - x) \right\} f_x(y)dy \\
+ \int_{\tau}^{\infty} \left\{ \lambda \bar{G}(y-x) + \int_0^{y-x} \Lambda(x+u)g(u)du + \bar{G}(\tau - x) \right\} f_x(y)dy - \lambda p_0 \\
= \mathcal{L}^0(x) + \bar{G}(\tau - x) [1 - \lambda F_x(\tau)]
$$

(10)

$\mathcal{L}^\tau(x)$ is continuous and differentiable in $\tau$ under the assumptions of the model and

$$
\frac{d\mathcal{L}^\tau}{d\tau} = g(\tau-x)f_x(\tau) \left[ \frac{\lambda F_x(\tau) - 1}{f_x(\tau)} - \frac{\lambda}{r_G(\tau-x)} \right].
$$

As both $r_F$ and $r_G$ are nondecreasing, it can be shown that the expression in square brackets is monotone in the range where $\lambda F_x(\tau) - 1 > 0$. As it is also continuous, it has at most one change of sign. Moreover, $\lim_{\tau \to \infty} \Lambda^\tau(x) = \Lambda_0$ and $\lim_{\tau \to 0} \Lambda^\tau(x) = \Lambda_0 + 1$. So if (9) holds, $\Lambda$ is always above $\Lambda_0$, and it is optimal not to inspect. On the other hand, if (9) does not hold, there is a unique solution $\tau^*$; moreover, $\Lambda(\tau^*) < \Lambda_0$, and it is optimal to screen at $\tau^*$. □

The following corollary highlights a useful property of the solution and also provides help in the numerical evaluation of $\tau^*$.

**Corollary 1** Let $\bar{\tau}$ be the value that makes inspection at $\bar{\tau}$ indifferent to no inspection. Then

(i) $\bar{\tau} = F_x^{-1}(\lambda^{-1})$

(ii) $\tau^* > \bar{\tau}$.

*Proof*: The equality follows from (10). The inequality follows from $\lim_{\tau \to \infty} \Lambda^\tau(x) = \Lambda_0$ and $\lim_{\tau \to 0} \Lambda^\tau(x) = \Lambda_0 + 1$ and uniqueness. □

If in addition the density $f$ is log-concave, the determination of the optimal number of inspections $n$ is simplified by the following result. There is no need to compute the optimal times to determine the optimal number of inspections.
Theorem 2 Let \( f \) be log-concave and \( r_G \) be increasing; then, at an optimum, \( n \) is either 0 or \( \infty \). Moreover, \( n = \infty \) if and only if (9) is satisfied.

Proof: Define:

\[
\varphi(\delta, \tau) = \frac{\lambda F_x(\tau) - (\lambda - 1)F_x(\tau - \delta) - 1}{f_x(\tau)}, \quad \tau > 0, \quad \delta > 0 \tag{11}
\]

\[
\zeta_0(\tau) = \frac{\lambda \tilde{G}(\tau - x)}{g(\tau - x)} \tag{12}
\]

Suppose that that \( n \) examinations are planned at times \( \tau_1, \ldots, \tau_n \). As in the proof of Theorem 1, an additional examination is desirable if there is a value satisfying: \( \varphi(\tau, \tau + \tau_n) - \zeta_0(\tau) = 0 \).

Next, note that under log-concavity \( \partial \varphi(\delta, \tau)/\partial \tau \geq 0 \). In fact, at \( \lambda = 1 \), \( \varphi = -1/rF(\tau) \) which increases in \( \tau \). Moreover,

\[
\varphi(\delta, \tau) = \lambda \frac{F_x(\tau) - F_x(\tau - \delta)}{f_x(\tau)} - \frac{\tilde{F}_x(\tau - \delta)}{f_x(\tau)};
\]

the first term is increasing in \( \tau \) for fixed \( \delta \) as a direct consequence of log-concavity (see Barlow and Proschan, 1965), so that \( \partial \varphi(\delta, \tau)/\partial \tau \) is increasing in \( \lambda \), whence the result.

Therefore:

\[
0 < \lim_{\tau \to \infty} [\varphi(\tau, \tau) - \zeta_0(\tau)] < \lim_{\tau \to \infty} [\varphi(\tau, \tau + \tau_n) - \zeta_0(\tau)].
\]

From (11), \( \lim_{\tau \to \infty} \varphi(\tau, \tau) = \lim_{\tau \to \infty} (\lambda - 1)/f_x(\tau) \), so a solution exists by Theorem 1. Iterating gives the required conclusion.

Conversely, assume (9) does not hold. Then the one-screen continuation problem has no solution, the risk is always decreasing in \( \tau \) and always above \( A_0 \), so that no single-inspection schedule is preferable to no inspection. To show that no second inspection \( \tau_2 \) exists that makes \( \tau \) satisfy the first order conditions, consider (8) and note that the left hand side.

\[
\zeta(\delta, \tau) = \frac{(\lambda - 1)G(\tau + \delta - x) - \lambda G(\tau - x) + 1}{g(\tau - x)} \tag{13}
\]
is increasing in \( \delta \). Hence no inspecting is preferable to any schedule. \( \square \)

If both units are aging, it is natural to expect optimal interval between inspections to decrease. The following theorem establishes this fact when aging is modelled by log-concavity.

**Theorem 3** If \( f \) is log-concave then at an optimum \( \tau_{i+1} - \tau_i < \tau_i - \tau_{i-1} \).

**Proof:** Let \( \varphi \) and \( \zeta \) be defined by (11) and (13) respectively. In Theorem 2 it was established that \( \partial \varphi / \partial \tau > 0 \). If \( g \) is log-concave, a similar argument leads to \( \partial \zeta / \partial \tau < 0 \). Now rewrite (8) as

\[
\zeta(\delta_{i+1}, \tau_i) = \varphi(\delta_i, \tau_i) \quad i \geq 1
\]

(14)

Then,

\[
\frac{\partial \delta_{i+1}}{\partial \delta_i} = \frac{\partial \varphi(\delta_i, \tau_i)}{\partial \delta} \bigg|_{\delta=\delta_i} = \frac{f_x(\tau_{i-1})g(\tau_i - x)}{f_x(\tau_i)g(\tau_{i+1} - x)} > 0
\]

(15)

and

\[
\frac{\partial \delta_{i+1}}{\partial \tau_i} = \frac{\partial \varphi(\delta_i, \tau_i)}{\partial \tau} \bigg|_{\tau=\tau_i} = \frac{-\partial \zeta(\delta_{i+1}, \tau_i)}{\partial \tau} \bigg|_{\delta=\delta_i+1} > 0
\]

Then, at an optimum, \( \delta_i > \delta_{i-1} \) implies \( \delta_{i+1} > \delta_i \). Next, we need to show that if for some \( i \), \( \delta_i > \delta_{i-1} \) then there is an \( k \geq i \) such that no solution \( \delta_{k+1} \) exists to the optimal continuation, contradicting \( n = \infty \). From log-concavity, \( f \) and \( g \) are unimodal. Call the modes \( m_f \) and \( m_g \).

Take \( i \) so that \( \tau_{i-1} \geq m_f \vee (m_g + x) \). Let \( \delta_{i+1} = r \delta_i \), \( r > 1 \). Then,

\[
\frac{\partial^2 \zeta(\delta_i, \tau_i)}{\partial \delta^2} \bigg|_{\delta=\delta_{i+1}} = (\lambda - 1) \frac{g'(\tau_i + \delta_{i+1} - x)}{g(\tau_i - x)} < 0
\]

\[
\frac{\partial^2 \varphi(\delta, \tau_i)}{\partial \delta^2} \bigg|_{\delta=\delta_i} = -(\lambda - 1) \frac{f_x(\tau_i - \delta_i)}{f_x(\tau_i)} > 0
\]

Moreover, \( \varphi(0, \tau) < 0 \) and \( \zeta(0, \tau) > 0 \). Thus, using optimality and the above partial derivatives
and boundary inequalities,

\[ \zeta(\delta_{i+1}, \tau_i) = \varphi(\delta_i, \tau_i) = r \varphi(\delta_{i-1}, \tau_{i-1}) > \varphi(\delta_{i-1}, \tau_{i-1}) = r \zeta(\delta_i, \tau_{i-1}) = r \zeta(\delta_i, \tau_i) \]

In turn, this entails \( \delta_{i+1} > r \delta_{i+1} \), so that the sequence of increments increases without bound.

From convexity, \( \varphi \) does as well. Also, as a function of \( i \),

\[ \zeta(\delta_{i+1}, \tau_i) \leq \frac{\lambda}{r \sigma(\tau_1)}. \]

Therefore there is a sufficiently large \( k \) such that \( \varphi(\delta_k, \tau_k) > \lambda / r \sigma(\tau_1) \), so that a solution to (8) does not exist, contradicting \( n = \infty \). Thus times between examinations must decrease. \( \square \)

The next result gives uniqueness of the solution.

Theorem 4 Let \( f \) be \( PF_2 \) and \( g \) decreasing. Let \( \tau_1^* \) be the optimal first inspection, and \( \tau \) the schedule obtained by applying the optimality conditions (8) to an arbitrary initial \( \tau_1 \). Then:

(a) if \( \tau_1 > \tau_1^* \) there is an \( i \) such that \( \delta_i > \delta_{i-1} \);

(b) if \( \tau_1 < \tau_1^* \) then there is an \( i \) such that \( \delta_i < 0 \).

Proof:

From (8),

\[ \frac{d \delta_2}{d \tau_1} \left| \frac{\partial \zeta(\delta, \tau_1)}{\partial \delta} \right|_{\delta = \delta_2} = \frac{\partial \varphi(\delta, \tau_1)}{\partial \delta} \bigg|_{\delta = \tau_1} + \frac{\partial \varphi(\tau_1, y)}{\partial y} \bigg|_{y = \tau_1} - \frac{\partial \zeta(\delta, y)}{\partial y} \bigg|_{y = \tau_1} > 0 \]

Then, \( d \tau_2 / d \tau_1 = 1 + d \delta_2 / d \tau_1 > 1 \) and using (15),

\[ \frac{d \delta_3}{d \tau_1} = \frac{\partial \delta_3}{\partial \delta_2} \frac{d \delta_2}{d \tau_1} + \frac{\partial \delta_3}{\partial \tau_2} \frac{d \tau_2}{d \tau_1} > \frac{\partial \delta_3}{\partial \delta_2} \frac{d \delta_2}{d \tau_1} = \frac{f_x(\tau_1) g(\tau_2 - x) \, d \delta_2}{f_x(\tau_2) g(\tau_3 - x) \, d \tau_1}. \]
By induction, it can than be shown that:

\[
\frac{d\delta_i}{d\tau_1} \geq \frac{f(\tau_1)g(\tau_2 - x) \ d\tau_2}{f(\tau_{i-1})g(\tau_i - x) \ d\tau_1} \quad i > 2.
\]

From the above, \(d\delta_i/d\tau_1\) can be made arbitrarily large in \(i\). The results follows. \(\square\)

Theorem (4) can also be used to implement a binary search of the optimal \(\tau_1\) for fixed \(\lambda\). Therefore, under the assumption of Theorem (4) an algorithm for the choice of the optimal policy is the following.

- Choose an initial \(\tau_1\).

- Using Lemma (3), compute the optimal continuation until either
  
    (i) \(\tau_i < \tau_{i-1}\) or
    
    (ii) \(\tau_{i+1} - \tau_i > \tau_i < \tau_{i-1}\) or
    
    (iii) \(\tau_i > T\), where \(T\) is a fixed horizon;

- If (i) increase \(\tau_1\) and restart; if (ii) decrease \(\tau_1\) and restart; if (iii) move to the next step.

- Repeat the previous steps until \(E\{S\} = p_0\).

This section is closed by a simple example.

**Example.** Assume that the initiating events are generated by a Poisson process with rate \(\mu_G\), so that \(X_i\)'s are independent exponentials. Assume also that the failure times of the standby unit are exponential with rate \(\mu_F\). Then the optimal policy is periodical, that is \(\tau_i = \delta i\) for some \(\delta\). Using Lemma (3), for fixed \(\lambda\), if condition (9) is satisfied, the optimal \(\delta(\lambda)\) is the unique solution of:

\[
\frac{\lambda}{\lambda - 1} \left( \frac{1}{\mu_G} + \frac{1}{\mu_F} \right) = \frac{1}{\mu_G} e^{-\mu_G \delta} + \frac{1}{\mu_F} e^{-\mu_F \delta}
\]

Moreover, \(\Lambda\) is independent of \(x\) so that the \(\lambda\) satisfying the constraint can be determined based
on the following Lagrangean, obtained solving (1):

\[ \Lambda = \frac{\mu_G + \mu_F}{\mu_F} A \tau. \]

where \( \tau = \{ \delta(\lambda), 2\delta(\lambda), \ldots \} \).

5. Comments

This paper develops method for finding optimal inspection times for stand-by units. The fundamental elements of the solution are a Fredholm integral equation for the risk function — that solves the problem of adapting the policy to new information, and the recursive conditions (8) — offering a simple and intuitive relationship between successive inspection times. The relationship depends only on a cost-modified version of the variation of the two c.d.f.'s involved. If both units are aging attractive properties of the solution arise.

I wish to conclude by pointing to two useful directions for extending the results obtained in this paper. First, one could account for error probabilities of the inspection procedure. An integral equation and recursive conditions can be derived for tests subject to error, but the methods of proof of Section 4 apply only to the very special case of a nearly flat density \( g \). For further details on test errors in simpler models, see Parmigiani (1993a).

Second, in some application it may be important to relax the assumption that \( Y \) and \( U \) are independent. For example, failures of the main unit and failure of the stand-by may be subject to the same external causes (other than time which is already incorporated in the model). The ideas developed in this paper can be used to build models with dependent failures.
REFERENCES


