INSPECTIONS FOR LOG-CONVEX FAILURE DISTRIBUTIONS

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ABSTRACT

This paper discusses the theory of optimal scheduling of inspections for the early detection of failures when the distribution of the failure time is log-convex. The interest of the results is the intrinsic importance of log-convex distributions as well as in the analysis of failure distributions not fully known. It is shown that, within the class of log-convex densities, times between successive inspections increase with time. The case of exponential failures with unknown failure rate is explored in detail, with discussion of the issues of learning and of sensitivity to the specification of the prior distribution.

KEYWORDS: Optimal inspection schedule, log-convex failure distributions, unknown failure rate.
1. INTRODUCTION

Scheduling inspections times for the detection of system failures is a common problem in reliability; see for example [3]. This paper extends classic results of optimal inspection theory to the class of log-convex failure distributions. This extension is motivated on the one hand by the intrinsic importance of log-convex distribution; on the other hand by the fact that log-convex distribution arise as a very convenient way of modeling cases in which the failure distribution is only partially known. Of paramount importance is the special case of a failure distribution known only to be exponential, but with unknown failure rate.

Study of inspection models begun with Barlow Hunter and Proschan [1], and has been continued by many authors (see [10], [5], [6], [8] and references therein for some of the recent developments). The basic assumptions of inspection models can be conveniently summarized as follows.

(i) A system is monitored for failures. The state of the system is a continuous time process with two states, labeled safe functioning and failure;
(ii) The system starts at time 0 in the functioning state; the time to failure is random;
(iii) Information on the state of the system can be obtained only by a suitable inspection procedure;
(iv) The inspection procedure is free of error;
(v) There is a cost associated with each unit of time elapsed between the failure and its detection;

(vi) Inspections terminate upon detection of the failure.

The decision problem to be solved consists of choosing the inspection times so to minimize the s-expected cost. Before discussing the extension considered in this paper, it is useful to review the notation.

**Notation**

$Y$ system failure time, a random variable

$\Lambda$ unknown s-parameter(s), random vector with values in $\Omega \subseteq \mathbb{R}^p$

$g(\cdot | \lambda)$ conditional pdf of $Y$ given $\Lambda = \lambda$

$\pi(\cdot)$ pdf of $\Lambda$

$\Pi(\cdot)$ Cdf of $\Lambda$

$f(\cdot)$ pdf of $Y$

$k$ cost of one inspection

$c$ cost of one unit of elapsed time between failure and detection

$\tau_i$ time of the $i$-th inspection

$\tau = \{\tau_i\}_{i=0,1,2,...}$ infinite sequence (or schedule) of inspection times

$X^\tau$ time of the last inspection performed, a random variable

$I^\tau$ number of inspections performed, a random variable

$R(\tau)$ expected cost (or risk) of using the schedule $\tau$
2. LOG-CONVEX FAILURE DENSITIES

The family of log-convex failure distribution is interesting in a variety of applications. In the context of this paper, one important reason for considering it is the following. In many cases, the assumption of a fully known failure distribution—maintained in [1], as well as in the subsequent developments—can be unrealistic. One interesting question is how the classic Barlow, Hunter and Proschan [1] solution to the inspection problem is affected by uncertainty regarding the failure distribution. A natural way to approach this question is to assume that the parametric family of the failure distribution is known, but some of the parameters are unknown. In some cases, the distribution of these parameter may be known [9]; in others it can assigned based on expert opinion, following the Bayesian approach.

From the initial assumptions of the model, the cost suffered if the schedule \( \tau \) is adopted is \( kI^\tau + c(X^\tau - Y) \). The problem is choosing the sequence \( \tau \) that minimizes the expected cost, or risk:

\[
R(\tau) = kE(I^\tau) + cE(X^\tau - Y).
\]  

(1)

Expectations are taken with respect to everything that is unknown at decision time, that is \( Y \) and \( \Lambda \).
The risk function (1) can be rewritten in terms of the inspection times as:

\[ R(\tau) = \int_{\Omega} \left[ \sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} [k(i + 1) + c(\tau_{i+1} - y)]g(y|\lambda) \right] d\Pi(\lambda) \]  

(2)

Assuming that the summation in square brackets is finite, and using the fact that

\[ f(y) = \int_{\Omega} g(y|\lambda)d\Pi(\lambda), \]  

(3)

the risk function can be written as:

\[ R(\tau) = \sum_{i=0}^{\infty} (k(i + 1) + c\tau_{i+1})[F(\tau_{i+1}) - F(\tau_i)] - cE(Y). \]  

(4)

Interestingly, the risk can be computed based on the marginal failure density of \( Y \) alone. Partial differentiation of (2) with respect to each element of \( \tau \) gives:

\[ \tau_{i+1} - \tau_i = \frac{F(\tau_i) - F(\tau_{i-1})}{f(\tau_i)} - \frac{k}{c} \quad i = 1, 2, \ldots \]  

(5)

These equations define recursively the optimal inspection schedule once the optimal value of \( \tau_1 \) is given. This reduces the infinite dimensional optimization to a unidimensional problem, namely the choice of the optimal initial inspection \( \tau_1 \). This problem was solved by Barlow, Hunter and Proschan [1] in the special case of log-concave failure densities.
In particular, they proved that the inspection intervals decrease, consistent with the fact a device aging according to such failure density wears out with time.

Importantly, the optimal policy arising from the marginal failure density when parameters are random can differ in very important ways from the policies arising from the conditional distributions when the parameters are assumed known. In particular, the class of log-concave densities is not closed under mixtures (see [2]). Thus the theory of Barlow, Hunter and Proschan [1] may apply to every element in the family of conditional failure distributions, and not to the corresponding marginal failure distribution. This provides motivation for extending the analysis to classes of distributions that arise as mixtures of failure distributions commonly considered. Of particular interest is the class of log-convex densities, which contains, among others, all marginal failure densities resulting from exponential or Pareto distributions with unknown parameters.

Consider the following motivating example. A system fails according to an exponential distribution with unknown failure rate. Suppose an inspection is made. If the system is still working, it is known from the memoryless property of the conditional failure distribution that no physical deterioration took place; on the other hand there is new information about the value of the parameter: Having observed no failures makes one "more optimistic" about the lifetime of the device. Therefore, it should be optimal to wait longer before inspecting again. The main result in the next section shows that this is true, and more generally, that all log-convex failure densities yield increasing times between
inspections.

Before moving to this, it is convenient to summarize properties of log-convex densities. See [3] and [7] for further discussion.

**Property 1** Let $\Delta > 0$. Then $f$ is log-convex if and only if the ratio $f(y - \Delta)/f(y)$ is nonincreasing in $y$.

**Property 2** $f$ is log-convex if and only if, for every $\Delta > 0$, the ratio $[F(y + \Delta) - F(y)]/f(y)$ is nondecreasing in $y$.

**Property 3** The ratio $[F(y) - F(y - \Delta)]/f(y)$ is nonincreasing in $y$ for every $\Delta > 0$.

**Property 4** (DFR) The failure rate: $h(y) = f(y)/[1 - F(y)]$ is nonincreasing in $y$.

For positive random variables, under suitable regularity conditions (see [7]) the converse also holds.

**Property 5** Let $g(y|\lambda)$ be log-convex as a function of $y$ for every fixed $\lambda$ in $\Omega$. Then $f(y) = E\{g(y|\lambda)\}$ is log-convex.

One practical importance is that the class of log-convex densities contains all mixtures of exponential densities with unknown parameter, regardless of the prior distribution. However, mixtures of exponentials do not exhaust the class of log-convex densities. In fact, mixture of exponentials are completely monotone [4], and complete monotonicity is
a stronger requirement than log convexity, the latter involving conditions on the first two
derivatives only.

3. INCREASING TIMES BETWEEN INSPECTIONS

**Theorem 1** The optimal inspection schedule for a log-convex failure density has increasing interchecking times.

*Proof:* The argument is by contradiction: in summary, if interchecking times begin
to decrease, they never stop decreasing; also, they will decrease at a geometrical rate;
this implies an infinite expected loss. Since an optimal strategy with finite risk exists, as
guaranteed by Theorem 1 in [1], interchecking times have to be increasing.

Let \( \delta_i = \tau_i - \tau_{i-1} \) be the interchecking times associated with the sequence \( \{\tau_i\}_{i=1,2,...} \).

From (5), the optimality conditions are:

\[
\delta_{i+1} = \frac{F(\tau_i) - F(\tau_i - \delta_i)}{f(\tau_i)} - \frac{k}{c}. \tag{6}
\]

Suppose that \( \delta_i \leq \delta_{i-1} \). Then, by Property 3:

\[
\delta_{i+1} - \delta_i = \frac{F(\tau_i) - F(\tau_i - \delta_i)}{f(\tau_i)} - \frac{F(\tau_{i-1}) - F(\tau_{i-1} - \delta_{i-1})}{f(\tau_{i-1})} \leq 0 \tag{7}
\]

Hence, if \( \delta_i \leq \delta_{i-1} \), then \( \delta_{i+1} \leq \delta_i \).
Let now \( r < 1 \), and consider the relation

\[
\int_0^{r\Delta} f(y - t) \, dt \leq r \int_0^{\Delta} f(y - t) \, dt. \tag{8}
\]

Formula (8) is true if and only if:

\[
\int_0^{r\Delta} f(y - t) \, dt \leq r \left[ \int_0^{r\Delta} f(y - t) \, dt + \int_{r\Delta}^{\Delta} f(y - t) \, dt \right], \tag{9}
\]

that is:

\[(1 - r) \int_0^{r\Delta} f(y - t) \, dt \leq r \int_0^{\Delta} f(y - t) \, dt. \tag{10}\]

Now, the right hand side is not less than \( r(1 - r)\Delta f(y - r\Delta) \), whereas the left hand side is not greater than the same quantity, because \( f \) is decreasing. So (8) holds.

Next, \( f(y - \Delta)/f(y) \) is nonincreasing by Property 1. Hence, for \( \Delta > 0 \), \( \tau_i > \tau_{i-1} \) and \( \tau_i - r\Delta > 0 \) it follows:

\[
\int_0^{r\Delta} \frac{f(\tau_i - t)}{f(\tau_i)} \, dt \leq r \int_0^{\Delta} \frac{f(\tau_i - t)}{f(\tau_i)} \, dt \leq r \int_0^{\Delta} \frac{f(\tau_{i-1} - t)}{f(\tau_{i-1})} \, dt \tag{11}
\]

Integrating and substituting \( \delta_{i+1} = \Delta \) gives:

\[
\frac{F(\tau_i) - F(\tau_i - r\delta_i)}{f(\tau_i)} \leq r \frac{F(\tau_{i-1}) - F(\tau_{i-1} - \delta_{i-1})}{f(\tau_{i-1})} \tag{12}
\]
Now suppose that interchecking times start decreasing, that is that we can find an $i$ such that $\delta_i - r\delta_{i-1} \leq 0$. Then, by the optimality conditions and the above inequality:

$$
\delta_{i+1} - r\delta_i = \frac{F(\tau_i) - F(\tau_i - \delta_i)}{f(\tau_i)} - \frac{rF(\tau_{i-1}) - F(\tau_{i-1} - \delta_{i-1})}{f(\tau_{i-1})} - \frac{k}{c}(1 - r) \\
\leq \frac{F(\tau_i) - F(\tau_i - r\delta_{i-1})}{f(\tau_i)} - \frac{rF(\tau_{i-1}) - F(\tau_{i-1} - \delta_{i-1})}{f(\tau_{i-1})} \leq 0. 
$$ (13)

Hence, if $\delta_i \leq r\delta_{i-1}$, then $\delta_{i+1} \leq r\delta_i$; that is, if interchecking times decrease, they decrease at a geometrical rate. As an optimal strategy with finite risk exists, it is sufficient to show that if interchecking times start decreasing, the expected loss associated with the sequence is infinite. So suppose, for a contradiction, that there is a $k$ such that $\delta_k = r\delta_{k-1}$, with $r < 1$. Then $\delta_{k+1} - r\delta_k \leq 0$ and $\tau_{k+1} \leq \tau_k + r\delta_k$. Thus $\tau_{k+2} \leq \tau_{k+1} + r\delta_{k+1} \leq \tau_k + r\delta_k + r^2\delta_k$.

Iterating gives:

$$
\tau_{k+n} \leq \tau_k + \delta_k \sum_{i=1}^{n} r^i,
$$

so that for all $k$:

$$
\lim_{n \to \infty} \tau_n \leq \tau_k + \delta_k \sum_{i=1}^{\infty} r^i = \tau_k + \frac{\delta_k}{1 - r} < \infty. 
$$ (14)

It follows from the DFR property of log-convex densities that $\forall y > 0, 1 - F(y) > 0$; hence:

$$
P \left\{ Y > \tau_k + \frac{\delta_k}{1 - r} \right\} = 1 - F \left( \tau_k + \frac{\delta_k}{1 - r} \right) > 0
$$
and there is a positive probability that the failure is not detected. This entails an infinite expected loss, and the desired contradiction is reached. □

Log-convexity is sufficient, but it is not necessary, to obtain increasing times between checks. To see this, consider a distribution for which log-concavity holds everywhere except in the interval \((0, \epsilon)\). Then, as long as the ratio \(k/c\) is such that \(\tau_1 > \epsilon\), times between checks will be decreasing.

These results can provide the basis for future interesting developments. Two important examples are: Inspection of stand-by systems, in which the loss function might be in terms of the probability that the standby fails to work when required, rather than in terms of the detection delay; Sequential inspections, in which the information gathered inspecting a system is used to improve the performance in future inspection problems.

4. SENSITIVITY ANALYSIS

If elicitation of information about the time to failure is difficult, it is worthwhile to evaluate the practical influence of different prior specifications.

First, it is important to carefully weigh the consequences of the choice of a family of prior distributions on the rate of convergence of the tail of the resulting marginal. The latter has a noticeable impact on the optimal schedule, and it may be that two prior specifications that look similar in the middle of the distribution imply marginals that are different enough to have tails with increasing interchecking times in one case and
decreasing in the other. Secondly, changes in location of the marginal distribution not affecting the scale will tend to have a smaller effect than changes of scale. Finally, any practical implementation of inspection policies will have a minimum feasible unit of time (say hours or days) that can be adopted in practice. The study of the sensitivity of the risk to changes in the prior has to account for the fact that, in some instances, the approximation to the continuous time hypothesis may not be very accurate. When this is the case, rounding of the first few inspection times may amplify (or absorb) the effect of differences in the prior specifications.

These considerations suggest that it is reasonable to specify a prior distribution, compute the resulting policy, and then give a numerical evaluation of the sensitivity of the risk associated to that specific policy and prior. When a parametric family of failure densities is adopted, a practical, computationally inexpensive and easy to interpret quantitative evaluation of sensitivity is the derivative of the risk with respect to prior hyperparameters, evaluated at the optimum. This constitutes a good indication of a problematic situation, and may suggest a more careful assessment of the prior information, or the choice of some more complex hierarchical model—which would be cumbersome in a case of low sensitivity.

To illustrate this technique, consider the case of an exponential distribution with a conjugate Gamma prior. This case can be worked out analytically and will help sharpen the focus on the problem. It is also very commonly used in applications.
The results include the partial derivatives of the risk with respect to both hyperparameters of the Gamma prior, and the case in which it is desired to keep the marginal expected value of the lifetime fixed in the sensitivity evaluation. Other constrained sensitivity evaluations can be carried out in an analogous fashion.

Let \( \pi(\lambda) = b^a \lambda^{a-1} e^{-b\lambda} / \Gamma(a) \) for \( \lambda > 0 \), where \( \Gamma(a) \) is the Gamma function. The marginal distribution of \( Y \) is:

\[
f(y) = \int_0^\infty \lambda e^{-\lambda y} \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} d\lambda = \frac{a}{b} \left( \frac{b}{b + y} \right)^{a+1}
\]

for \( y > 0 \), which is a Pareto density. In particular, then, from (2) the risk is:

\[
R(\tau) = \sum_{i=0}^\infty [k(i + 1) + c\tau_{i+1}] \left[ \left( \frac{b}{b + \tau_i} \right)^a - \left( \frac{b}{b + \tau_{i+1}} \right)^a \right] - \frac{cb}{a - 1}.
\]

Now:

\[
\frac{dR}{da} = \sum_{i=1}^\infty \frac{dt_i}{da} \frac{\partial R}{\partial t_i} + \frac{\partial R}{\partial a}.
\]

If \( \tau \) is optimal, then \( \partial R / \partial \tau_i = 0 \) for every \( i \), so that \( dR/da = \partial R/\partial a \). Hence

\[
\frac{dR}{da} = \sum_{i=0}^\infty [k(i + 1) + c\tau_{i+1}] \left[ \left( \frac{b}{b + \tau_i} \right)^a \log \left( \frac{b}{b + \tau_i} \right) - \left( \frac{b}{b + \tau_{i+1}} \right)^a \log \left( \frac{b}{b + \tau_{i+1}} \right) \right] + \frac{cb}{(a - 1)^2}
\]

\[12\]
\[
\frac{dR}{db} = \sum_{i=0}^{\infty} \left[ k(i + 1) + c\tau_{i+1} \right] \left( \frac{b}{b + \tau_i} \right)^{a-1} \left( \frac{\tau_i}{(b + \tau_i)^2} - \frac{b}{b + \tau_{i+1}} \right)^{a-1} \left( \frac{\tau_{i+1}}{(b + \tau_{i+1})^2} \right) - \frac{c}{a - 1}.
\]

Next, let the marginal expected lifetime \(E(Y) = \frac{b}{a-1}\) be fixed, and consider the sensitivity of the risk with respect to the choice of \(a\). Note that \(db/da = E(Y)\), so that:

\[
\frac{dR}{da} = \sum_{i=0}^{\infty} \left[ k(i + 1) + c\tau_{i+1} \right] \left[ \frac{dF(\tau_{i+1})}{da} - \frac{dF(\tau_i)}{da} \right],
\]

where:

\[
\frac{dF(\tau_i)}{da} = \left( \frac{b}{b + \tau_i} \right)^a \log \left( \frac{b}{b + \tau_i} \right) \left[ \left( \frac{b}{b + \tau_i} \right)^{a-1} \frac{\tau_i}{(b + \tau_i)^2} E(Y) \right]
\]

\[
= \left( \frac{b}{b + \tau_i} \right)^{2a} \frac{\tau_i}{b + \tau_i} \log \left( \frac{b}{b + \tau_i} \right).
\]

Note that the programming of these functions constitutes a trivial and very inexpensive addition if a program already computing the risk is available.

5. SUMMARY AND CONCLUSION

In summary, this paper achieved two main goals. The first is to indicate, based on a Bayesian predictive approach, how to address inspection problems in presence of uncertainty regarding the failure distribution. The second is to derive a counterpart of
the classic result of Barlow, Hunter and Proschan [1] for the case of log-convex failure densities, arising naturally as a marginal failure distribution. As specification of a prior distribution may be difficult or controversial, practical indications and analytic results for sensitivity analysis are also given.

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