A MODEL FOR SEGMENTATION
AND ANALYSIS OF NOISY IMAGES

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A Model for Segmentation and Analysis of Noisy Images

By Valen E. Johnson

Abstract. I propose a statistical model for image generation that provides automatic segmentation of images into intensity-differentiated regions and facilitates the quantitative assessment of uncertainty associated with identified image features. The model is specified hierarchically within the Bayesian paradigm and at the lowest level in the hierarchy a Gibbs distribution is employed to specify a probability distribution on the space of all possible partitions of the discretized image scene. An important and novel feature of this distribution is that the number of partitioning elements, or image regions, is not specified a priori. At higher levels in the hierarchical specification, random variables representing emission intensities are associated with regions and pixels. Observations are assumed generated from exponential family models centered about these values.

Key Words. Bayesian inference, Gibbs distributions, image restoration, hierarchical models, random partitions, emission computed tomography, hexagonal arrays.

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1. INTRODUCTION

Visual displays of spatial data in the form of images play an important role in many scientific investigations. In part, this is because images permit researchers to utilize human pattern recognition skills in the identification of objects within image scenes. Unfortunately, images are less useful in providing quantitative information about the qualitative features discerned by human observers. For instance, differences in emission intensities between distinct image regions cannot easily be extracted visually from a grey-level image; indeed grey scales are often adjusted to enhance contrast and may represent highly nonlinear maps of the estimated intensities. Moreover, estimating region volumes in three-dimensional image reconstructions is a difficult task using available graphic facilities, and uncertainty about identified image features is difficult to assess on the basis of a single estimate of the image scene.

As an example in medical imaging, consider single photon emission computed tomography (SPECT). SPECT images represent intensity maps of metabolic activity and are used in the diagnosis and treatment of cancer. When examining a SPECT image, a physician’s initial interest typically lies in the identification of tumors or other abnormalities. Upon tentative identification of such regions, estimation of region volumes and emission intensities are critical in staging disease and planning treatments. Furthermore, because of statistical fluctuations that are inherent in the collection of SPECT data, uncertainty regarding the presence or absence of a region (i.e. tumor) must also be assessed.

This paper describes a statistical model for image generation that facilitates quantitative image analysis. The proposed model is parameterized so that estimates of the true scene provide automatic segmentations of the image into intensity-differentiated regions. Through techniques like Gibbs sampling, this parameterization also provides a simple mechanism for estimating the posterior distributions of region volumes and mean intensities, and offers the potential for evaluating the posterior probability of the existence of a region at a particular image location.

The model is specified hierarchically within the Bayesian paradigm. At the lowest level of the hierarchy, a Gibbs distribution is employed to specify a probability distribution on the space of all possible partitions of the discretized image scene. A novel feature of this distribution is that the number of partitioning elements, or image regions, is not assumed known a priori. Other segmentation models (e.g. Liang et al 1991) require that the number of regions be specified prior to image restoration. Since such prior information is seldom actually available, allowing the number of regions to vary within the model framework is an important practical feature of the prior distribution proposed. Also, by specifying the partition model at the most abstract level in the hierarchy, it is possible to evaluate the properties of the partition model independently
of the properties of the specific system used to obtain image data.

In the second level of the model, a random variable representing emission intensity is associated with each partitioning element or region. Individual pixel intensities are assumed to be drawn from a distribution centered around the region intensities in the third stage, and in the final stage of the hierarchy observations are assumed to be generated from an exponential family distribution centered on the pixel intensities.

A further advantage of specifying the model in a hierarchical framework is that observations at pixels can often be modeled as being conditionally independent given their pixel intensities. This is an important feature when modeling non-Gaussian observations since Gibbs priors specified directly in terms of local differences between pixel intensities are restricted in the form of the likelihood functions employed (Besag 1974). For example, the conditional distributions of sites within a Gibbs field cannot be of the gamma type if the conditional expectation is linked in a sensible way to the values of neighboring sites. Thus, gamma likelihood functions result in conditional posterior distributions that are difficult to analyze. Additionally, the assumption of conditional independence in the final stage of the model often permits the normalization function to be written explicitly as a function of second and third stage hyperparameters. Thus, the marginal posterior distributions of parameters that determine local smoothness within image regions can often be estimated directly from image data.

Related to the model proposed here is an extensive literature illustrating the use of Gibbs distributions as models for image scenes. A seminal paper in this area is that of Geman and Geman (1984) in which the Gibbs sampler was proposed as a mechanism for sampling from Gibbs distributions and simulated annealing was used to obtain maximum a posteriori (MAP) estimates of image scenes. Also proposed in this article were several image restoration models that utilized discrete line sites to segment finite alphabet images. The line sites were introduced into the graph system associated with the Gibbs distributions to prevent smoothing across boundaries that separate homogeneous regions. These models proved to be useful in segmenting four to six level images, but were found to be computationally unmanageable for continuous valued images (Johnson et al 1991a). Johnson et al (1991b) extended the Geman and Geman models to include continuous valued line sites and proposed a variation of Besag's iterated conditional modes (ICM, Besag 1986) algorithm to obtain point estimates of image scenes for continuous valued images. The continuous line sites were found to improve the sampling properties of the posteriors, but were less successful in imposing smoothness constraints on the shapes of the estimated boundaries. Related line site models have recently been proposed by Leahy and Yan (1991) and Gindi et al (1991).

A shortcoming of models that incorporate lines sites is that segmentations of images do not follow
readily from estimated configurations of line sites. Because neighborhood systems associated with Gibbs distributions are typically limited in size, it is difficult to ensure that line sites connect with one another to form closed regions. Even when a configuration of line sites does partition an image, determining the partitioning sets from the line sites is a nontrivial task that is best not repeated iteratively within the Gibbs sampling framework.

Alternative formulations to line site models that encourage local smoothing within regions while permitting sharp boundaries between regions involve the use of bounded potential functions. Gibbs models that employ bounded potential functions have been proposed, for example, by Geman and McClure (1985, 1987) and Green (1990), with the latter also describing techniques for obtaining MAP estimates in the image reconstruction setting (i.e. estimation of image scenes based on projection data rather than direct observation). However, these models do not provide a mechanism for segmenting images and are therefore ill-suited for the tasks addressed here.

In addition to the articles already cited, the reader interested in reviewing properties and definitions of Gibbs distributions as they relate to the image model proposed below may refer to Besag (1972), Derin et al (1984), and Derin and Elliot (1987).

2. MODEL DESCRIPTION

Let the true scene be discretized into either a rectangular or hexagonal array of pixels, and denote this array by $\Xi = \{\xi_{ij}\}$. Assume that the true scene is comprised of an unknown number of intensity-differentiated objects. With each possible configuration of objects associate a partition of $\Xi$, where a partition is defined here as any collection of sets of connected pixels in which each pixel appears in one and only one set. A set of pixels will be considered connected if it is possible to move from any pixel in the set to any other pixel in the set without leaving the set. When $\Xi$ is defined on a rectangular array, movement between pixels that either touch at a corner or share a common side is permitted, while for hexagonal arrays only movement between pixels that share a common side is permitted.

In order to define a probability distribution on the class of all partitions, assign to every pixel in the array $\Xi$ an integer such that all pixels in each partitioning set are assigned the same integer, and each partitioning set is associated with a distinct integer. The particular integers chosen are otherwise arbitrary. These integers are called region identifiers and the array of region identifiers is $R = \{r_{ij}\}$. A Gibbs distribution can be defined on the array of region identifiers by specifying a neighborhood system, cliques, and potentials. A neighborhood system on a graph $\Xi$ is defined to be any collection of
subsets $G = \{ G_\xi, \xi \in \Xi \}$ such that $\xi \notin G_\xi$, and $\xi_1 \in G_\xi$ if and only if $\xi_2 \in G_\xi$. A clique is defined as any subset of $\Xi$ in which every element is a neighborhood of every other element. Denote the set of cliques by $C$.

A portion of a simple neighborhood system defined on a hexagonal lattice is depicted in Figure 1. The subgraph in Figure 1a shows the six first order neighbors of the central pixel (which is itself not a member of the neighborhood). Up to arbitrary rotations, the two clique types that result from this neighborhood structure are depicted in Figure 1b.

figure 1 here

Given the array $\Xi$ and a neighborhood system $G$, a Gibbs distribution on the region identifier array $R$ must have a density function expressible in the form

$$\pi(R) = \frac{1}{Z} \exp\{-U(R)\}, \quad U(R) = \sum_{C \in C} V_C(R). \quad (2.1)$$

In this expression, $Z$ is called the partition function and is a constant independent of $R$. $U(R)$ is called the energy function, and the functions $V_C$ are called potentials. Importantly, the potential $V_C$ can depend only on components of $R$ that belong to the subscripted clique $C$. An important consequence of this form of the energy function is that the conditional distribution of any component of the system, say $r_{ij}$, given the values of all other components of $R$, depends only on changes in potential functions that result from changes in the value of site $r_{ij}$.

To specify the prior density on the class of partitions, it therefore suffices to specify a neighborhood system (and by implication an associated clique system) and potential functions. For our purposes, the neighborhood system is defined as the entire graph, so that every region identifier is in the neighborhood of every other region identifier. This neighborhood system would generally make the implied distribution computationally intractable since the conditional distribution of any site could well depend on every other site. However, the nonzero potential functions employed in this model are easily computed and the large neighborhood system in fact poses little computational difficulty.

The Gibbs distribution on the class of image partitions can be used to model prior notions regarding likely configurations of objects in the true scene. Three such properties are modeled here. First, configurations having large numbers of regions are discouraged. In noisy images, this type of constraint is needed to prevent individual pixels from assuming completely arbitrary values, a problem that makes maximum likelihood estimation unattractive in many settings (e.g. Vardi, Shepp, and Kaufmann 1985). Second, irregular object shapes are discouraged, although the extent to which they are penalized depends on the nature of the true scene. Third, configurations in which objects are not connected are prohibited. In two-dimensional image
restoration this condition may not always be sensible, although in three-dimensional image reconstruction it almost always is. Of course, the constraint can be dropped when it is deemed inappropriate.

The first potential type, designed to restrict the number of regions, is a function of the entire graph. The particular form of the potential depends on the nature of the image scene and the anticipated number of objects in the scene. If little prior information regarding the number of regions is available, a possible choice for this potential might be \( V_1(\mathbf{R}) = \alpha K \), where \( K \) represents the number of distinct region identifiers in the graph and \( \alpha \) is an arbitrary hyperparameter. The effect of this potential is to impose a constant penalty of \( \exp(\alpha) \) on the formation of each new region. Alternatively, when the number of regions in the image scene is known to be relatively small, a potential of the form

\[
V_1(\mathbf{R}) = \alpha K^2 - \alpha K
\]  

might be used. This potential imposes a penalty of \( \exp(2\alpha K) \) for the formation of the \((K + 1)^{th}\) region. Thus, the penalty for new regions increases linearly with the number of existing regions. Similarly, a cubic potential can be used to impose a quadratic penalty around some \textit{a priori} estimate of the number of image regions. Note that the computational burden associated with this clique is not problematic despite the fact that the clique contains the entire graph. Criteria for selecting appropriate values of \( \alpha \), along with other hyperparameter values, are discussed in Section 3.

The second type of potential is employed to impose regularity on the shapes of regions. One possibility for this type of potential was proposed by Derin and Elliot (1987) in the context of binary image segmentation and assigns an energy of, say, \( \phi \) to each pair of neighboring pixels not assigned the same region identifier. Such potentials impose local regularity, but in some cases may not be appropriate since the increase in energy associated with long smooth boundaries is also quite large. This can cause the prior distribution to concentrate around scenes having only one or two distinct regions. An alternative on a hexagonal lattice is to assign a potential of \( \phi \) to each set of seven nearest neighbor pixels if those exterior pixels that are assigned the same region identifier as the center pixel are not connected within the clique when the center pixel is removed. For example, the configuration depicted in Figure 2a has zero energy according to this potential, while the configuration in Figure 2b is assigned energy \( \phi \). A similar potential can be assigned to each \( 3 \times 3 \) square in a rectangular lattice. On the hexagonal lattice, this potential is denoted \( V_2(C_7) \), with \( C_7 \) indicating the particular seven pixel clique under consideration. Note that there are numerous configurations having both large and small regions that are not penalized under this form of potential.

The final potential type is included to prevent a region from splitting into two disconnected partitions.
To understand the need for such a potential, suppose that a partitioning set has the shape of a dumbbell, and that the connecting "bar" is one pixel wide. Then when updating any of the pixels in the bar, a change in the value of the given region identifier would separate the dumbbell into two distinct regions, requiring that all region identifiers in one of the two segments be changed. However, such changes violate the Markovian property of the Gibbs distribution, and so pixels in the bar are not permitted to change. This is accomplished by assigning infinite potentials to changes in region identifiers that result in disconnected regions. Like the constraint on the number of regions, the clique associated with this potential is the entire graph. (Technically, infinite potential functions pose a difficulty in the Gibbs formulation since they violate the positivity constraint used in the Hammersely-Clifford theorem (Besag 1974). However, in this case an aperiodic, irreducible Markov chain with equilibrium distribution and transition probabilities given by the implied Gibbs conditional distributions can be constructed using the Metropolis algorithm (e.g. Metropolis et al 1953, Ripley 1987).)

To gain insight into the distribution of partitions, it is worthwhile to consider the conditional distribution used to update region identifiers within the Gibbs sampler. Suppose then that the unlabeled region identifier in the portion of a hexagonal lattice depicted in Figure 3 is to be updated, and that all region identifiers not shown are currently assigned the value 1. In this case, the value of the region identifier assigned to the unlabeled pixel cannot result in a region separation, and so the third potential function is zero for all possible updates. The $V_2$ potentials that result when this pixel is assigned the values 1 and 2 are 0 and $\phi$, respectively (the 2's in the perimeter of the $C_7$ clique centered at the upper left 2 would not be connected if the unlabeled pixel was assigned the value 2 and the center pixel removed). Since there are currently two region identifiers in the lattice, using the $C_4$ potential given in (2.2) results in an increase in energy of $4\alpha$ if a new region, say region 3, is formed. Hence the relative probabilities that the unlabeled pixel is assigned the values 1, 2, and 3 are $\exp(0)$, $\exp(-\phi)$, and $\exp(-4\alpha)$.

Given a configuration of region identifiers, or equivalently a partition of the image into objects, the next stage in the model associates with each partitioning set an intensity parameter. Individual pixel intensities within regions are assumed to be drawn from a distribution centered around this value. Typically, the form of this distribution is taken to be conjugate to the distribution used to model the observed pixel values in the final stage of the model. In the final stage of the model, an exponential family distribution is assumed for the generation of observations at individual pixels.

As an example, consider a Poisson-gamma model for the observations and pixel intensities. In this case,
let $\mu_k$ denote the mean intensity for pixels in partitioning set $k$, let $\Lambda = \{\lambda_{ij}\}$ denote the array of pixel intensities, and let $Y = \{y_{ij}\}$ denote the array of observed Poisson counts at individual pixels. Then given the region identifiers, the hierarchical model for image generation can be expressed

$$y_{ij} \mid \lambda_{ij} \sim \text{Poisson}(\lambda_{ij}),$$

$$\lambda_{ij} \mid r_{ij} = k, \mu_k, \nu \sim G(\mu_k, \nu).$$  \hspace{1cm} (2.3)

In this parameterization, $G(\mu, \nu)$ denotes a random variable with density function

$$g(\lambda; \mu, \nu) = \frac{1}{\Gamma(\nu)} \left( \frac{\nu \lambda}{\mu} \right)^\nu \exp\{-\nu \lambda / \mu\}. \hspace{1cm} (2.4)$$

This particular intensity model forms the basis for many emission computed tomography systems in medical imaging, and is used as an example in Section 4.

3. ESTIMATION

The hierarchical specification of the image generation model using a Gibbs prior on possible partitions permits efficient sampling from the posterior distribution. Through sampling, the posterior mean, the MAP estimate, and local maxima from the image scene can be determined (e.g. Geman and Geman 1984, Besag 1986). Additionally, the posterior distributions of region volumes and means can be estimated.

The mechanism used for sampling is the Gibbs sampler proposed in the image context by Geman and Geman (1984) and subsequently developed for more general applications by Tanner and Wong (1987) and Gelfand and Smith (1990). The basic requirement for implementing the Gibbs sampler is that full or reduced conditional distributions for all unknown quantities be available. In the hierarchical specification of our model, the full conditional distributions for the intensity related parameters given the region identifiers may be derived using standard results in Bayesian inference. For example, under the Poisson-gamma model described above, the full conditional distributions for the intensity parameters may be expressed

$$\lambda_{ij} \mid y_{ij}, r_{ij}, \mu_{r_{ij}}, \nu \sim G \left( \frac{\nu \lambda_{ij}}{(\mu_{r_{ij}} / \nu) + 1}, y_{ij} + \nu \right),$$

$$p(\mu_k \mid \Lambda, R) \propto \exp \left[ -\sum_{r_{ij} = k} \frac{\nu \lambda_{ij}}{\mu_k} - \sum_{r_{ij} = k} \nu \log(\mu_k) \right],$$

$$p(\nu \mid \Lambda, R, \mu) \propto \pi(\nu) \prod_{ij} \frac{\nu \lambda_{ij}^{\nu}}{\Gamma(\nu)} \exp \left( -\frac{\nu \lambda_{ij}}{\mu_{r_{ij}}} \right). \hspace{1cm} (3.1)$$

Note that a vague prior is assumed for $\mu$ and that the prior density for $\nu$ has been denoted $\pi(\nu)$.
The conditional distribution of $R$ is similarly straightforward and follows a Gibbs distribution. In the Poisson-gamma model, the conditional distribution of the unlabeled pixel in Figure 3, say pixel $\xi_{ij}$, follows a multinomial distribution with relative probabilities

$$\Pr(r_{ij} = 1) \propto \exp(-\nu \lambda_{ij}/\mu_1 - \nu \log(\mu_1)),$$

$$\Pr(r_{ij} = 2) \propto \exp(-\nu \lambda_{ij}/\mu_2 - \nu \log(\mu_2) - \phi),$$

$$\Pr(r_{ij} = 3) \propto \exp(-\nu \lambda_{ij}/\mu_3 - \nu \log(\mu_3) - 4\alpha).$$ (3.2)

It should be noted that the subscripts in (3.2) are pixel dependent and hence slightly ambiguous. In this expression, $\mu_1$ denotes the mean intensity of the partitioning set that includes $\xi_{ij}$ and all other pixels currently assigned region identifier 1. Likewise for $\mu_2$ and $\mu_3$. Thus, all three means are sampled when updating the unlabeled region identifier, although conceptually all components of $\mu$ could be sampled prior to this update. Computationally, sampling the means for all partitioning sets that arise when updating the region identifiers is feasible due to the existence of sufficient statistics for exponential family models. Of course, an obvious approximation to this sampling scheme is simply to associate the mean intensities with the region identifiers themselves, particularly for those regions that contain a large number of pixels.

In exponential family models with conjugate priors, alternative sampling schemes utilizing reduced conditional distributions are also possible. For example, in the Poisson-gamma model, if $\theta$ denotes a vector with components

$$\theta_k = \frac{\mu_k/\nu}{\mu_k/\nu + 1},$$

then the reduced conditional distributions for the transformed variables $\theta_k$ have beta densities expressible as

$$\theta_k \mid R, Y \propto \theta_k^{\alpha-1}(1-\theta_k)^{\beta-1}, \quad \alpha = 1 + \sum_{r_{ij}=k} y_{ij}, \quad \beta = -1 + \sum_{r_{ij}=k} \nu.$$ (3.3)

Similar reductions apply to the region identifiers and scale parameters. For example, in Figure 3 the unlabeled region identifier can be updated according to relative probabilities

$$\Pr(r_{ij} = 1) \propto \theta_1^{\nu r_{ij}} (1 - \theta_1)^{\nu},$$

$$\Pr(r_{ij} = 2) \propto \exp(-\phi) \theta_2^{\nu r_{ij}} (1 - \theta_2)^{\nu},$$

$$\Pr(r_{ij} = 3) \propto \exp(-4\alpha) \theta_3^{\nu r_{ij}} (1 - \theta_3)^{\nu}.$$ (3.4)

and the reduced conditional distribution of $\nu$ is

$$p(\nu \mid Y, R, \mu) \propto \pi(\nu) \prod_i \frac{\Gamma(\mu_{rij} + \nu)}{\Gamma(\nu)} \left( \frac{\mu_{rij}}{\mu_{rij} + \nu} \right)^{\nu r_{ij}} \left( \frac{\nu}{\mu_{rij} + \nu} \right)^{\nu}. \quad (3.5)$$
Use of reduced conditional distributions greatly increases the computational efficiency of the resulting sampling schemes, and in the case of the Poisson-gamma model was noted to increase by a factor of 30 the number of region identifiers that changed in each update of $R$. Presumably, this decreased dependence between successive iterates in the Gibbs sampler implies that fewer iterations are required to reach equilibrium.

Unfortunately, the conditional distributions of the hyperparameters in the Gibbs prior model for $R$ are not available directly to the intractability of the partition function. This problem is alleviated through the specification of the model in a hierarchical framework since the region model can be studied independently of the intensity model. One approach to studying the region model is to simulate partitions of the image array using the Gibbs sampler, and to then select values of the hyperparameters based on the appearance of the simulated partitions. By varying $\phi$, the shape of regions can be modulated, and by varying $\alpha$ the distribution of the number of regions can be adjusted. This approach is examined in more detail below.

4. AN EXAMPLE

To illustrate the performance of the model, consider the restoration of a standard phantom used in nuclear medicine, the 3-D Hoffman brain phantom (Hoffman et al 1990). This phantom contains regions assigned one of three distinct intensity levels, corresponding to white matter, grey matter, and ventricles or background. For our purposes, we will assume that background and ventricles have different emission rates, so the phantom employed here actually consists of regions assigned one of four intensity levels. The modified phantom is depicted in Figure 4a. Regions corresponding to white matter were assigned a mean rate of 20, grey matter regions mean rate 10, ventricles mean rate 2, and background regions mean rate 0. Individual pixel intensities were drawn from gamma distributions with these means and scale parameter $\nu = 100$. Poisson observations were then generated at each pixel according to the sampled pixel intensities. The true pixel intensities and a Poisson observation of the image are displayed in Figures 4b and 4c. The goal of our analysis is to segment the observed image, estimate the volumes and mean emission rates for each region, and to assess the uncertainty associated with these estimates. However for conciseness, quantitative results are cited only for the darkened region depicted in Figure 4d.

figure 4 here

An initial step towards estimating the posterior distribution of the image was identification of plausible values for the hyperparameters $\alpha$ and $\phi$ appearing in potentials $V_1$ and $V_2$. As stated above, these hyperparameters were chosen by sampling from the prior distribution of the region identifier array. Hyperparameter
values that lead to sampled images having approximately the anticipated distributions of shape and number of regions were chosen.

In implementing the sampler, several procedures were used to overcome the effects of starting values. First, numerous initial values having a broad range of region numbers were used, and initial values having both too few and too many regions were employed. Equilibrium conditions were assumed to obtain when the distribution of the number of regions was approximately the same for the different starting values, and when region births and deaths appeared approximately equally likely. Based on the appearance of sampled images, parameter values $\alpha = 0.25$ and $\phi = 3.0$ were selected. Images based on these values typically had between 19 and 25 regions, and two sampled images are depicted in Figures 5a and 5b.

figure 5 here

These hyperparameter values were used in the estimation of the posterior distribution of the observed image depicted in Figure 4c. Initial estimates of the image were obtained by grouping the observed pixel counts into six equally divided intensity intervals and then separating the pixel groups into connected regions. This resulted in an initial estimate of the region identifier array having 1134 elements. To speed convergence of the algorithm from this initial estimate, bordering regions were subsequently combined deterministically during early iterations of the sampler when the net energy of the system could be reduced through such combination. In each iteration, the reduced conditional distributions described in equations (3.3-5) were successively used to update region identifiers, region means, and the smoothing parameter $\nu$. The prior taken for $\nu$ was $G(100, 100)$ (the actual value being 100), and the order of updating region identifiers was again random. Sampled images from the posterior distribution appear in Figures 6a and 6b.

figure 6 here

Figure 7 depicts the results obtained in estimating the posterior distribution of the area and mean of the darkened region in Figure 4d. The true area of this region was 140 pixels and the mean intensity was 20. The histogram estimates for the posterior distributions of these parameters were obtained using values obtained in iterations 500-1000, and are similar in shape to those obtained in iterations 1000-5000.

figure 7 here

A histogram estimate of the posterior distribution of the smoothing parameter $\nu$ is shown in Figure 8. The prior density (unnormalized) is also shown for purposes of comparison. Note that the likelihood function peaked somewhat below the true value of the parameter, despite the fact that essentially 18,944 observations are available for its estimation. This downward bias is typical of restorations using this model.
and probably results from errors in the estimation of the region identifiers. However, the effects of this bias do not appear severe, and the small effects that do occur tend to result in overly conservative estimates of the posterior precision of region means and volumes.

figure 8 here

5. DISCUSSION

The proposed image model offers a framework for accomplishing a number of image analytic tasks not mentioned above. One such task involves the use of high resolution background information to improve estimation of object locations in comparatively low resolution images. For example in medical imaging, magnetic resonance images can be cross-correlated with SPECT images; in geological applications, locations of surface fault lines may be available when estimating subterranean fault surfaces. The proposed model permits inclusion of such information in a number of ways ranging from simply fixing region identifiers when they are known precisely to modifying the prior distributions on subsets of region identifiers when only imprecise information is available.

Another extension of the model involves testing for the existence of distinct regions within images. Again referring to SPECT, the posterior probability for the existence of a tumor at a given location within an image might be estimated by examining the proportion of sampled images in which a distinct region was present at the hypothesized location. In such applications, it seems reasonable to locally modify the potential associated with region formation to reflect the prior odds for the existence of a tumor, and to make the formation of a region in the area of interest independent of the total number of distinct region identifiers present in the remainder of the image.

The proposed model can also be modified in a straightforward fashion for 3-D application. Two lattice structures are appealing for use in defining neighborhood structures in three dimensions, the cubic lattice and lattices yielding optimal sphere packings (e.g. Sloane 1988). All of the potential functions suggested in the two-dimensional setting extend immediately to three dimensions.

A drawback of the model is that it ignores systematic trends in intensity variation within regions. Modeling this feature would require fitting generalized linear models to region means as functions of pixel location within the image. However, noise properties of many images might make such procedures impractical, particularly in smaller regions.

Finally, the Poisson-gamma model described for image restoration can be extended to image reconstruction using results described in Shepp and Vardi 1982 and Johnson et al 1991b. Such models are particularly
important in nuclear medicine and are the topic of extensive ongoing research.

REFERENCES


Figure legends

Figure 1. a) First order (nearest neighbor) neighborhood in hexagonal lattice. Note that the center hexagon is itself not a member of the neighborhood. b) Cliques associated with first order neighborhood. Any rotation of these cliques within the lattice structure also represents a clique in the first order neighborhood.

Figure 2. a) Configuration not penalized by $V_2$, b) configuration penalized by $V_2$. In (b), note that perimeter 2's are not connected within the clique without center 2.

Figure 3. Hypothetical configuration of region identifiers.

Figure 4. a) Hoffman phantom. This image represents a slice from a $64^3$ bitmap of a 3-D brain phantom. The original bitmap was converted to a $128 \times 148$ hexagonal lattice, and then converted back to a $512 \times 512$ rectangular lattice for display. Statistical modeling and processing were performed on the hexagonal array. b) A realization of the Hoffman phantom in which pixel intensities varied around the region means according to a $G(\lambda, 100)$ distribution. The background intensities are rate 0, the interior ventricles have mean rate 2, the grey matter mean rate 10, and the perimeter white matter mean rate 20. c) A Poisson observation of (b). d) The black region (lower left of the image center) is used to illustrate the quantitative properties of the image model in Section 4. In this and all other images presented, the slight artifacts near the edges of regions and the image boundary result from the conversion algorithm used to transform between hexagonal and rectangular arrays.

Figure 5. Sampled images from the Gibbs prior for the region identifier matrix. The colors associated with each partitioning set were chosen randomly, and the initial value for this image consisted of a $5 \times 4$ checkerboard pattern having 20 distinct region identifiers. Both images are from the same sample path. a) The $1000^{th}$ sampled image. This image has 24 distinct regions. b) The $20,000^{th}$ sampled image. This image has 22 distinct regions.

Figure 6. Samples from the posterior distribution of the image depicted in Fig. 3c. a) $500^{th}$ sampled image. b) $5000^{th}$ sampled image.

Figure 7. Histogram estimates of the posterior distribution of the region mean and volume of the highlighted region in Fig. 3d. The histograms were generated from iterations 500-1000 of the same sample path. a) Histogram of region intensities. The actual value was 20. b) Histogram of region area in hexagonal pixels. The actual value was 140.
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4c) A Poisson observation of (4b).

4d) The black region (lower left of the image center) is used to illustrate the quantitative properties of the image model in Section 4. In this and all other images presented, the slight artifacts near the edges of regions and the image boundary result from the conversion algorithm used to transform between hexagonal and rectangular arrays.
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6b) 5000th sampled image.