RESERVE PRICING AT SINGLE-OBJECT AUCTIONS

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Abstract

At oral auctions it is frequently observed that auctioneers not only employ a "standard" reserve policy but, in addition, they use a retractable reserve. Specifically, if a bidder withdraws before the retractable reserve is reached then the auctioneer does not retain the item but, rather, sells the item to the highest bidder. In this paper we demonstrate that a retractable reserve allows the auctioneer to exploit both affiliation of the valuations and risk aversion.

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I. Introduction

A central issue in the theory of auctions concerns the differentiation of auction schemes by their expected revenue to the seller. The fundamental result within this area of research is the Revenue Equivalence Theorem — within the independent private values (IPV) model with risk neutral bidders a broad class of auction schemes are identical in terms of the expected revenue they generate for the auctioneer.\footnote{See Harris and Raviv (1981), Riley and Samuelson (1981) and Myerson (1981).} Included in this group are the English, Dutch, first price and second price auctions. Replacing risk neutrality with risk aversion induces a revenue separation; first price and Dutch auctions dominate second price and English auctions.\footnote{See Riley and Samuelson (1981) and Maskin and Riley (1984).} Within the general symmetric (GS) model with risk neutral bidders we know that English auctions weakly dominate second price auctions which in turn weakly dominate first price auctions.\footnote{See Milgrom and Weber (1982).}

In practice, it is well known that ascending bid auctions are extremely prevalent (see Cassady (1967, pg. 56)), especially when government auctions (almost entirely sealed-bid first price) are excluded. Intuitively, ascending bid auctions should have a great appeal to the seller for three reasons. First, if bidders’ valuations or value estimates are affiliated then the observable exit points of low-valued bidders will influence the bidding behavior of those who remain active. This is the essence of the argument by Milgrom and Weber (1982, referred to subsequently as MW) for the weak dominance of the English auction. Second, unlike other types of auctions, the ability to observe bids prior to declaring a winner provides useful information to the auctioneer; information which he can exploit to raise the price paid for the item. Third, ascending bid auctions provide the auctioneer with greater strategic latitude than other schemes (elaborated below). The last two reasons are explored in this paper.

One of the implications of the Revenue Equivalence Theorem is that there is no useful information in the bid sequence at an ascending bid auction. This result follows directly from the fact that, with the exception of the actual valuations drawn (which are independent of one another), the auctioneer knows the characteristics of each bidder. In previous work (Graham and Marshall (1987) and Graham, Marshall, Richard (1987, 1989)) it was shown that if the distributional identities of heterogeneous bidders are unknown to the auctioneer he can learn valuable information from the bid sequence about the distributional origin of the highest bid. In fact, phantom bidding (also referred to as “lift-lining”, “trotting”, or “running” — see Cassady (1967, pg. 213–214)), which depends upon information from the observed bids, generates higher expected revenue for the seller than the use of an ex ante fixed reserve. This is one explanation for why phantom bidding is a commonly observed phenomena at ascending bid auctions (see Cassady (1967, pg. 213))

Traditional modeling of English auctions involves the use of a single non-retractable reserve price, whether it is fixed ex ante or it is an optimal phantom bid by the auctioneer. By non-
retractable we mean that if the reserve is greater than the bid of the highest valued bidder then the auctioneer retains the item.

However, auctioneers who sell by means of ascending bid auction often retract phantom bids. We have obtained evidence of this practice at auctions of horses, wine, and antiques. Experienced bidders are aware of this practice. In fact, it is typically the case that auctioneers will have two phantom bids — a non-retractable reserve and then a higher retractable bid. In essence, the latter is a "dummy" bid by the auctioneer.

An important issue with retractable phantom bids concerns the price at which the item will be sold if the auctioneer is "caught" with the high bid. This issue is typically addressed in an auction catalog via the resolution of "bidding disputes". To understand the flexibility of the ascending bid auction suppose the rules of the auction are such that in the event of a dispute (i.e. the auctioneer's retractable phantom bid has induced the exit of the last active bidder from the bidding), the last active bidder is obligated to pay the amount of their final bid. In such a case the ascending bid auction has been transformed into something like a first price auction where the auctioneer has the added advantage of learning about bidders via the bid sequence (unlike a standard sealed bid first price auction). Alternatively, if the dispute rules specify that the last active bidder must pay the bid of the last underbidder (assuming this is verifiable) then the ascending bid auction is one with characteristics of both an English and a first price auction. In this paper, unless otherwise indicated, when an auctioneer retracts a phantom bid the last active bidder pays the non-retractable phantom bid associated with the second highest bid. A strong justification for this specific choice of retraction will emerge from our analysis.

A simple numerical example may illuminate these distinctions. Suppose the auctioneer observes the second to last bidder exit the bidding at $60. If the auctioneer uses only a non-retractable reserve policy he can either (i) award the item to the last active bidder for $60 or (ii) invoke a reserve of, say, $80 where the item will be retained by the auctioneer if the last active bidder withdraws before $80 or, alternatively, the item will be sold for $80 if the last active bidder does not withdraw. On the other hand, the auctioneer could use both a non-retractable and a retractable reserve. Suppose that in response to a second to last bid of $60 the auctioneer invokes a non-retractable reserve of $70 and a retractable reserve of $90. Three events are possible depending upon the withdrawal point of the last remaining bidder. If he withdraws below $70 then the auctioneer retains the item. If he is willing to bid above $90 then he wins the item for $90. However, if he withdraws between $70 and $90 then the item is awarded to him for the price of $70, the non-retractable reserve. Note that if the retractable reserve was set arbitrarily close to $70 the auction would be nearly indistinguishable from a "standard"

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4 At Arabian horse auctions the auctioneer will frequently identify an inexperienced bidder as the source of the retractable reserve (when the auctioneer is "caught"). At wine auctions we have been informed that it is common for an auctioneer to contact the last underbidder after an auction, inform him that the lot was not sold, and offer the lot to him. With respect to antiquities, witnesses in the trial of Ron Pook (U.S. vs Ron Pook, 1987) testified that when they were the last underbidder for an item auctioneers or consignors occasionally contacted them, informed them the item was not sold, and asked if they were still willing to consider the purchase. These observed phenomena suggest the importance of retractable reserves as a strategic device of an ascending bid auctioneer.

5 Retention of an item by the auctioneer with subsequent reauctioning of it at a later date is an activity similar to retraction of a phantom bid at a single object auction but requires dynamical game analysis. The former activity has been well documented in a paper by Ashenfelter (1989).
English auction. The same would be true if the retractable reserve was set extremely high, say at $1 billion. The auctioneer’s objective is to set the retractable reserve at an “intermediate” value such that the largest possible expected payment is extracted from the highest bidder.

In this paper we develop a framework for analyzing ascending bid auctions which accommodates (i) learning by the auctioneer via the bid sequence and (ii) both non-retractable and retractable phantom bids by the auctioneer. Our analysis is conducted within an affiliated symmetric private values (ASPV) model.

The results of our analysis for affiliated risk neutral bidders include the following. First, an auctioneer who is conducting an ascending bid auction increases expected revenue by making his non-retractable reserve a function of the bid sequence. Specifically, an ex ante fixed reserve is inferior in terms of expected revenue. Intuitively, the bid sequence conveys useful information to the auctioneer about the highest valued bidder — information which is ignored if a fixed reserve policy is employed.  

Second, due to the affiliation of the valuations, a first price auction generates lower expected revenue than English auctions (with either retractable or non-retractable reserves). This result plays an essential role in our analysis. It will be shown that the expected payment by a winning bidder (with valuation \( v \)) increases when the auctioneer’s pricing scheme is correlated with the way the conditional distribution of the second highest valuation (given \( v \)) shifts under affiliation. For first price auctions there is zero correlation which implies a reduction in the expected payment of the winning bidder.

When bidders are risk averse we obtain a rather striking result — an ascending bid auction with only a non-retractable reserve (function of the bid sequence) is dominated by one where, in addition, a retractable reserve is employed. The intuition for this result parallels the intuition for the expected revenue dominance of the first price auction in an IPV framework with risk averse bidders. Again, the retractable phantom bid induces “shading” (bidding less than ones valuation) by the bidders. Risk averse bidders, concerned with the potential forfeiture of a positive surplus, shade less than risk neutral bidders with identical valuations.

The use of retractable phantom bids at ascending bid auctions has an obvious advantage in light of this result. Specifically, by conducting an ascending bid auction the auctioneer can (i) exploit bidder affiliation (again, established earlier by MW) and (ii) can set a non-retractable reserve which optimally exploits the information contained in the bid sequence. This second point extends the analysis of MW who considered only ex ante fixed reserves. Furthermore, by adding a retractable phantom bid the auctioneer can increase revenue by exploiting the risk aversion of the bidders. The remarkable feature of this result is that ascending bid auctions can be transformed, via an expansion in the strategies available to the auctioneer, to exploit risk aversion while not suffering from the inability to exploit affiliation (like first price auctions).

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6 Throughout the paper, the terms “phantom bids” and “reserves” are used interchangeably.
7 This result is similar to the ones of Graham, Marshall, and Richard (1987,1989) but is derived within a different environment. These authors analyze a situation where the distributional identity of bidders is unknown to the auctioneer and the observed bid sequence conveys useful information to the auctioneer about the distributional identity of the winning bidder.
8 This result is a natural extension of MW’s Theorem 15 within the ASPV model. In itself, the ASPV model is less general than the GS model, but MW only consider fixed reserves.
The paper proceeds as follows. In Section II the assumptions of the model are specified. In addition, the rules of the ascending bid auction are defined. Section III contains the formal analytics of auctions with non-retractable reserves. "English" auctions with retractable reserves are investigated in Section IV. Further refinements are discussed in Section V. Concluding comments are offered in Section VI.

II. The Model

In establishing a framework for analysis our goal is to specify the simplest possible environment while retaining the essential features of observed behavior. First, we provide definitions of the auction schemes under consideration. Then all remaining assumptions are specified.

II.1 The Auctions

Before any actions are taken at the auctions the following information is common knowledge to all participants. Each of the N bidders are at bidding stations that have bidding buttons. A thermometer device with a dollar scale appears before the bidders. The thermometer initially starts at zero. A bidder is actively bidding as long as his button is depressed. Once a bidder removes his finger from the button he can not reenter the bidding (as seen subsequently, exit does not necessarily imply that a bidder is a loser). No information regarding the number of active bidders is publicly available to the bidders at any time. The auctioneer can pre-program the thermometer to increase in response to one and only one bid — the second highest.\(^9\) The thermometer always increases when at least two bidders are active. No useful information evolves during the course of the auction for the bidders. On the other hand, the information evolving during the auction is extremely useful for the auctioneer in terms of increasing expected revenue. However, the pre-programming constitutes a binding pre-commitment by the auctioneer as to how the information will be used. Within this informational framework we can now define a number of auction schemes.

*English auction.* The auctioneer can pre-program only a non-retractable reserve as a function of the second highest bid. At the moment the second to last bidder withdraws from the bidding either (i) the item is awarded to the last active bidder at the amount indicated on the thermometer or (ii) the thermometer continues to rise in response to the auctioneer's non-retractable phantom bid. In the latter case, if the thermometer stops while the bidder is still active then the bidder wins the item for the amount indicated on the thermometer. Otherwise, the auctioneer retains the item.

*First Price auction.* The auctioneer programs a non-retractable reserve as a function of the second highest bid. In addition, the thermometer continues to rise as long as one bidder is active.\(^10\) When the last bidder withdraws he wins the item for the amount shown on the

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\(^9\) It would be simple to wire the thermometer in such a way that the auctioneer would be able to observe lower level bids before taking any action (e.g. by displaying the number of active bidders). Our assumption that the auctioneer's reserve solely depends upon the second highest bid simplifies our analysis while retaining the informative nature of the bid sequence for the auctioneer.

\(^10\) Further discussion of this particular definition is offered in Section V.2.
thermometer, provided that amount exceeds the non-retractable reserve. If it does not, then the auctioneer retains the item.

**Retractable English auction.** The auctioneer can pre-program both a non-retractable reserve and a higher retractable one, both as functions of the second highest bid. For convenience, let \( \delta_1 \) be the non-retractable reserve and let \( \delta_2 \) be the retractable reserve. If the last active bidder withdraws below \( \delta_1 \) then the auctioneer retains the item. If the last active bidder withdraws between \( \delta_1 \) and \( \delta_2 \) then the bidder wins the item and pays \( \delta_1 \). If the last is willing to remain active above \( \delta_2 \) then the thermometer stops at \( \delta_2 \) and the bidder pays \( \delta_2 \) for the item. Note that the retraction to \( \delta_1 \) guarantees that the auctioneer has a finite \( \delta_2 \).\(^{11}\)

General Comments. First, continuous bid increments and the prohibition of reentry in the bidding by a bidder who has exited substantially simplify the analytics. These elements of our ascending bid auctions are identical to those of the English auction employed by MW. Second, the pre-programming of the reserves (phantom bids) prevents the auctioneer from revealing the number of active bidders. In practice, it is very difficult for a bidder at an ascending bid auction to know if he is bidding against the auctioneer alone or if he faces “true” competition. Certainly the withdrawal point of bidders less than the second highest is rarely, if ever, observed.\(^{12}\) The pre-programming of reserves preserves this realism. Third, the pre-programming of reserves implies that \( \delta_1 \) is verifiable. This is essential for the Retractable English auction. If \( \delta_1 \) were not verifiable then the auctioneer could assert any value for \( \delta_1 \) when a bidder withdraws between \( \delta_1 \) and \( \delta_2 \). In fact, his incentives are to claim that \( \delta_1 \) is arbitrarily close to the withdrawal point of the highest bidder if \( \delta_1 \) is not verifiable. Fourth, the participation of bidders who have valuations that are less than the lowest possible non-retractable reserve is not problematic in the absence of entry fees or bid preparation costs. As discussed later, there are randomization devices that the auctioneer can employ to encourage the participation of these bidders. Our results are robust to the introduction of such devices. Fifth, although the auctions above are described as ascending bid, since the auctioneer is the only agent who can extract information from the bid sequence all of the auctions can be implemented as sealed bid auctions (discussed further in Section V). To do this we must allow the auctioneer to submit sealed bid reserves that are functions of the second highest bid.

II.2 Notation and Assumptions

**Basics.** Each of the \( N \) bidders has the same von-Neumann Morgenstern utility function \( U \) where \( U(0) = 0 \) and \( U'(x) \geq 0 \) for \( x \in [0, \infty) \). The cases of risk neutral and risk averse are both discussed in this paper. For risk neutrality it is assumed that \( U(x) = x \). For risk aversion it is assumed that \( U''(x) < 0 \), which in turn implies that the function \( \frac{U(x)}{U'(x)} - x \) is monotone.

\(^{11}\) The credibility of \( \delta_1 \) as the non-retractable reserve is not an issue in a Nash equilibrium if the second highest bid can be verified ex post by the highest bidder. In any event, our retraction is “hard-wired”.

\(^{12}\) This point is important for affiliated valuations. The weak dominance of MW’s English auction over the second price stems from the observation by the two highest valued bidders of the withdrawal points of the N-2 lowest valued bidders. If only the second highest is observable then, within their model, the English and second price auctions are equivalent in terms of expected revenue. However, this is not true within our context if the second price auctioneer can only declare an ex ante fixed non-retractable reserve.
increasing and, hence, nonnegative. The derivation of our main results under risk aversion solely depend on this latter condition. The risk neutral auctioneer owns the single-object to be sold and has a zero personal valuation for the item.

Symmetry. Let \( V_1, ..., V_n \) be the valuations of the \( N \) bidders. The joint density of the valuations is assumed to be symmetric in its arguments which implies that the problem can be analyzed from the point of view of an arbitrary bidder. Let \( V \) be the valuation of this bidder and \( W \) be the highest valuation of his rivals where the marginal density of \( V \) and the conditional density of \( W \) are

\[
f(v) \quad \text{and} \quad h(w|v) \quad \text{on} \quad [0, \infty)
\]

Let \( A \geq B \) denote the two highest from \( (V_1, ..., V_n) \). Clearly \( A=\max(V,W) \) and \( B \) is the highest from the \( \min(V,W) \) and the second highest from rivals.

**Lemma 1:** If the density of \( (V_1, ..., V_n) \) is symmetric in its arguments, then the joint density of \( A \geq B \) is given by

\[
g(a,b) = nh(b|a) f(a) \quad \text{for all} \quad (a,b) \quad a \geq b \geq 0
\]

Proof: see Appendix.

**Affiliation.** Loosely speaking, we require that higher values of \( V \) shifts the distribution of \( W|V \) towards the right. A weaker version than that used by MW is employed here.

**Definition:** Weak Affiliation.

\[
\forall M \geq 0 \quad \int_0^M \frac{\partial}{\partial v} h(w|v) \, dw = \int_0^M \frac{\partial}{\partial v} \ln h(w|v) \cdot h(w|v) \, dw \leq 0
\]

Note that as \( M \to \infty \), the integral in (3) tends to zero for all \( v \). Hence, affiliation in MW, which implies that \( \frac{\partial}{\partial v} \ln h(w|v) \) increases with \( w \), entails weak affiliation.

**Technical Comment.** The notion of symmetry introduced here is essentially that of exchangeability in statistics (as discussed e.g. by Lindley and Smith (1972)). It follows that, under continuity assumptions, the joint density of \( (V_1, ..., V_n) \) can be expressed in the form

\[
f(v_1, ..., v_n) = \int \left[ \prod_{i=1}^n f(v_i | \alpha) \right] \cdot \nu(\alpha) \, d\alpha
\]

where \( f(\cdot|\alpha) \) is a univariate density function depending on a random vector \( \alpha \) and \( \nu(\alpha) \) is a density function for \( \alpha \). Hence, \( f(v) \) and \( h(w|v) \) in eqn. (1) can be represented as follows

\[
f(v) = \int f(v|\alpha) \cdot \nu(\alpha) \, d\alpha
\]

\[
h(w|v) = \frac{1}{f(v)} \int \frac{\partial}{\partial w} R^{n-1}(w|\alpha) \cdot f(v|\alpha) \cdot \nu(\alpha) \, d\alpha
\]
Any choice of \( f (v|\alpha) \) and \( \nu (\alpha) \) generates an exchangeable distribution for the \( v \)'s. Restrictions might be required in order to satisfy the weak affiliation condition (3), though the latter holds for the example in the Appendix.

**Bid and Reserve Strategies.** The bid strategies of the bidders are symmetric and denoted as \( \psi (v) \). The non-retractable reserve or phantom bid of the auctioneer is denoted as \( \delta_1 (b) \) while the retractable reserve or phantom bid is denoted as \( \delta_2 (b) \). It is assumed that \( \delta_2 (b) \geq \delta_1 (b) \geq b \). The equilibrium strategies are assumed to be continuous and monotone increasing. This assumption implies that the strategies have inverse functions which considerably facilitates the analysis.

**Price.** The price paid by the winner is a function solely of the two highest bids and the phantom bid strategies. Hence, when our typical bidder wins \((v > W)\) he pays

\[
p_\delta (\psi (v), \psi (W)) .
\]

The reference to the \( \delta \)'s will be omitted when it is inessential to our argument. Specific expressions for \( p_\delta (\cdot) \) will be discussed subsequently. Nevertheless, our bidder must submit a bid which exceeds the *non-retractable* reserve \( \delta (B) \) (or \( \delta_1 (B) \) when it is paired with a retractable reserve \( \delta_2 (B) \)). The inequality \( \psi (x) \geq \delta (\psi (W)) \) is conveniently rewritten as \( \lambda (x) \geq W \), with

\[
\lambda = \psi^{-1} \circ \delta^{-1} \circ \psi
\]

Such a function \( \lambda_i \) is paired with each reserve function \( \delta_i \) \((i=1,2)\). Note that \( \lambda \) is monotone and that \( \lambda (x) \leq x \). Hence, the inequality \( \lambda (x) \geq W \) also implies that \( x \geq W \). Defining \( x_* \) implicitly via \( \lambda (x_*) = 0 \), it follows that the expected utility to a typical bidder with valuation \( v \) who selects \( x \) is

\[
\Pi (x, v) = \begin{cases} 
\int_0^{\lambda (x)} U [v - p_\delta (\psi (x), \psi (w))] h (w|v) \, dw & x > x_* \\
0 & \text{otherwise}
\end{cases}
\]

Note that \( \Pi (x, v) \geq 0 \) for \( x > x_* \) and, hence, that under monotonicity assumptions the bidder always finds it profitable to participate if \( v > x_* \). Consequently, his reservation value always coincides with \( x_* \) which is itself determined by the auctioneer's reserve strategy. The auctioneer's expected revenue is given by

\[
P = \int_{x_*}^{\infty} \left[ \int_0^{\lambda (x)} \int_0^{p_\delta (\psi (x), \psi (w))} h (w|v) \, dw \right] f (x) \, dx
\]

**Comments.** First, the expression of \( h (w|v) \) in equation (5) implicitly assumes that all bidders participate in the auction regardless of their valuations. Nevertheless, bidders with valuations that are less than the lowest possible non-retractable reserve \((v < x_*)\) have a zero expected gain.
from bidding. Given their indifference between bidding and non-participation we presume that they bid their true valuations. Since this assumption is important for the existence of equilibrium strategies under affiliation\(^{13}\) we will subsequently propose incentive devices such that low valued bidders strictly prefer participation (Section V).

Second, it seems natural to analyze the auctions in terms of \( \psi \) and \( \delta \) functions. However, since bidders reply to \( \delta \) with their bid functions \( \psi \), the relevant functions for characterizing various auction schemes are the \( \lambda \) functions. Hence, our symmetric Nash equilibrium results are discussed in terms of \( \psi \)'s and \( \lambda \)'s and, in particular, the price functional in formula (6) is reformulated in terms of the \( \lambda \)'s (examples are given in Section III). Equation (8) then serves to determine the bidder's best response to given reserve functions, while equation (9) is used to evaluate the auctioneer's best response to a given bid function. Nash equilibrium pairs (or triples) of strategies constitute fixed points of the corresponding optimality conditions, assuming existence. Our monotonicity assumptions allow retrieval of the \( \delta \)'s from the \( \lambda \)'s when the former are needed e.g. for practical implementation.

III. Non-Retractable Reserves

In this section a theorem is established which illuminates the conflict between affiliation and risk aversion for the expected revenue of an auctioneer who can employ a non-retractable reserve that is a function of \('b'\). Our theorem is derived under the general pricing scheme introduced in Section II. It is then applied to the first price and (non-retractable) English auctions.

\textit{Theorem 1:}

Under the assumptions of Section II:

(i) The equilibrium bid strategy \( \psi \) is the solution of the identity

\[
U \left[ v - p(\psi(v)) \right] \cdot h(\lambda(v)) \cdot \lambda'(v) = \int_0^\lambda(v) \left[ v - p(\psi(v)) \right] \cdot p_1'(\psi(v), \psi(w)) \cdot \psi'(v) \cdot h(w|v) \, dw
\]

where \( p_1'(\psi(v), \psi(w)) = \frac{\partial}{\partial u} p(u, \psi(w)) \big|_{u=\psi(v)} \)

for \( v > x_* \) together with the initial condition \( \psi(x_*) = x_* \);
(ii) The corresponding expected return to the auctioneer is

\[ \bar{P} = n \int_{x_*}^{\infty} P'(v) \cdot [1 - F(v)] \, dv \quad \text{where} \quad P'(v) = \rho_1(v) \cdot \lambda(v) \cdot \lambda'(v) + \int_0^{\lambda(v)} p(\psi(w), \psi(w)) \cdot \frac{\partial}{\partial v} h(w|v) \, dw \]

for \( v > x_* \) together with \( P(x_*) = 0 \). The function \( \rho_1(v) \) is implicitly defined in the Appendix. It equals one if the bidders are risk neutral but is greater than one if bidders are risk averse.

Proof: see Appendix.

The function \( \rho_1 \) does not have a convenient analytic expression except under constant risk aversion when \( U(x) = x^\alpha \) for \( 0 < \alpha \leq 1 \) in which case \( \rho_1 \) is given by

\[ \rho_1(v) = 1 + \left( \frac{1}{\alpha - 1} \right) \cdot \left( 1 - \frac{\psi(v)}{v} \right) \]

In the theorem the two terms which comprise \( P'(v) \) are extremely important. They do not vary independently. However, in general, the first term captures the effect of risk aversion while the second captures the effect of affiliation. Hence, Theorem 1 provides useful insight on the effect of bidders' risk aversion and (weak) affiliation on a broad range of auction schemes with non-retractable reserves and, in particular, on English and first price auctions.

Consider first the risk aversion term only. The English auction, which sets \( p'_1 \equiv 0 \) \( (\Rightarrow \rho_1(v) \equiv 1) \), is clearly dominated by any auction scheme which incorporates components of high bid pricing — \( p'_1 > 0 \). (Note that high bid pricing does not imply the use of a first price auction). On the other hand we note that \( w < \lambda(v) \Rightarrow \delta \circ \psi(w) < \psi(v) \), wherefrom it can be shown under weak affiliation that

\[ \forall v > x_*, \quad \int_0^{\lambda(v)} \left[ \psi(w) - \delta \circ \psi(w) \right] \cdot \frac{\partial}{\partial w} h(w|v) \, dw \leq 0 \]

Hence, English auctions accommodate affiliation better than first price auctions. It follows that when both affiliation and risk aversion are present, the two factors in equation (8)

\[ \text{...} \]

14 Though the English and first price auctioneers will use different \( \lambda \) functions it is nevertheless the case that a first price auctioneer can always do at least as well as he would by using the \( \lambda \) that is optimal for a second price auction. Thus our dominance result.
work in opposite directions for first price and English auctions. This extends the result of MW who find, in a similar context, that first price and English auctions can not be ranked.

However, and equally important, equation (8) suggests immediately that it should be possible to design better mixes of first price and English auctions to best exploit the joint presence of affiliation and risk aversion — such as Retractable English auctions.

Special Case: English Auction with Risk Neutral Bidders

Additional analytic results can be obtained in this environment where for \( v \geq x^*_s, \rho_1(v) \equiv 1, \quad \psi(v) \equiv v \) and the price is given by \( \lambda^{-1}(\omega) \). The expected return to the auctioneer is then given by

\[
\bar{P} = n \int_{x^*_s}^{u} P(v) \cdot f(v) \, dv
\]

with

\[
P(v) = \int_{0}^{\lambda(v)} \lambda^{-1}(w) \cdot h(w|v) \, dw
\]

Taking advantage of Lemma 1, we find that

\[
\bar{P} = n \int_{x^*_s}^{\infty} \left[ \int_{x^*_s}^{v} u \cdot h(\lambda(u)|v) \cdot \lambda'(u) \, du \right] f(v) \, dv
\]

\[
= \int_{0}^{\infty} \lambda^{-1}(u) \cdot \left[ 1 - G_A(\lambda^{-1}(u)|u) \right] \cdot g_B(u) \, du
\]

The optimal \( \lambda^{-1}_1 \) is given by \( R^*(u) \) where

\[
R^*(u) = \arg \max_{R \geq u} R \cdot [1 - G_A(R|u)]
\]

Under the additional assumption of no affiliation \( \bar{P} \) in (16) simplifies further into

\[
\bar{P} = n \int_{0}^{\infty} \lambda^{-1}(u) \cdot [1 - F(\lambda^{-1}(u))] \cdot h(u) \, du
\]

The optimal \( \lambda^{-1} \) is then given by\(^{15}\)

\[
\lambda^{-1}(u) = R_* \quad \text{for} \quad u \leq R_* \quad \text{and} \quad \lambda^{-1}(u) = u \quad \text{for} \quad u \geq R_*
\]

\(^{15}\) This can only be interpreted as a limiting result since we require \( \lambda \) to be a function.
where
\[
R_* = \arg \max_R R[1 - F(R)] .
\]
(20)

It follows that the optimal reserve scheme is
\[
\delta(u) = R_* \text{ for } u \leq R_* \quad \text{and} \quad \delta(u) = u \text{ for } u \geq R_*
\]
(21)

independently of the pricing scheme, and
\[
\bar{P} = n \int_{R_*}^{\infty} [uf(u) + F(u) - 1] \cdot H(u) \, du
\]
(22)

Equation (22) is Riley and Samuelson's (1981) Proposition 1. Specifically, we reproduce their result within our framework and extend it in two ways. First, since our reserves can depend upon the bid sequence we have effectively demonstrated that non-retractable phantom bidding can not generate improvements in expected revenue in this case. Second, \(H(u)\) can be any distribution (the same for everyone) representing an arbitrary bidder's probability of winning if he submits \(\psi(v)\). It need not be restricted to be \(F^{n-1}\) as in Riley and Samuelson (1981).

IV. Retractable English Auctions

We now utilize the two reserve functions described in Section II. The retractable reserve \(\delta_2\) and the non-retractable reserve \(\delta_1\) are such that
\[
\forall v \geq 0 \quad \delta_2(v) \geq \delta_1(v) \geq v
\]
(23)

As we have already discussed, the auctioneer's behavior is best characterized in terms of the functions \(\lambda_i = \psi^{-1} \circ \delta_i^{-1} \circ \psi\). Let the domain of definition of \(\lambda_1\) be \([x_1^*, \infty)\) where \(x_1^*\) and \(x_2^* \geq x_1^*\) belong to the auctioneer's decision set. Their practical implementation is given by
\[
R_i^* = \delta_i(0) = \psi(x_i^*)
\]
(24)

Note that for \(x \geq x_2^*\) we have
\[
x \geq \lambda_1(x) \geq \lambda_2(x) \geq 0
\]
(25)

Our typical bidder with valuation 'v' who submits a bid \(\psi(x)\) can only win the item if
\[
\psi(x) \geq \delta_1 \circ \psi(W) \quad \text{i.e.} \quad \lambda_1(x) \geq W
\]
(26)
in which case his payment is
\[
\begin{align*}
\begin{cases}
\psi \circ \lambda_2^{-1}(W), & \text{if } 0 \leq W \leq \lambda_2(x) \\
\psi \circ \lambda_1^{-1}(W), & \text{if } \lambda_2(x) \leq W \leq \lambda_1(x)
\end{cases}
\end{align*}
\]
(27)
Hence the expected surplus for our typical bidder is zero for \( x < x_1^* \), and

\[
\Pi(x, v) = \begin{cases} 
\lambda_1(x) \int_0^x U \left[ v - \psi \circ \lambda_1^{-1}(w) \right] \cdot h(w|v) \, dw & \text{for } x_1^* \leq x \leq x_2^* \\
\lambda_2(x) \int_0^x U \left[ v - \psi \circ \lambda_2^{-1}(w) \right] \cdot h(w|v) \, dw & \text{for } x_2^* < x \\
+ \int_{\lambda_1(x)}^{\lambda_2(x)} U \left[ v - \psi \circ \lambda_1^{-1}(w) \right] \cdot h(w|v) \, dw & \text{for } x_1^* \leq x \leq x_2^* \\
\end{cases}
\]

(28)

As already mentioned we assume that bidders with valuations less than \( x_1^* \) have no incentive to deviate from truthful reporting, i.e. \( \psi(v) = v \) for \( v < x_1^* \). (See Section V for further discussion.)

We can now state the main theorem for the Retractable English auction.

**Theorem 2:**

Under the assumptions of Section II,

(i) \( x_1^* = x_2^* = R_1^* = R_2^* \) (say \( x^* \))

(ii) \( \psi(x_*) = x_* \) and for \( v > x_* \),

\( \psi(v) \) is the solution of the equation

\[
U \left[ v - \psi(v) \right] \cdot h(\lambda_1(v)|v) \cdot \lambda_1'(v) = \\
\left\{ U \left[ v - \psi \circ \lambda_1^{-1} \circ \lambda_2(v) \right] - U \left[ v - \psi(v) \right] \right\} \cdot h(\lambda_2(v)|v) \cdot \lambda_2'(v)
\]

so that \( \lambda_1(v) > \lambda_2(v) \Rightarrow \psi(v) < v, \text{ and} \)

(iii) for \( v > x_* \)

\[
P'(v) = \rho_2(v) \cdot v \cdot h(\lambda_1(v)|v) \cdot \lambda_1'(v) + \\
\int_0^{\lambda_2(v)} \delta_2 \circ \psi \frac{\partial}{\partial v} h(w|v) \, dw + \int_{\lambda_1(v)}^{\lambda_2(v)} \delta_1 \circ \psi \frac{\partial}{\partial v} h(w|v) \, dw
\]

(30)

where the function \( \rho_2(v) \) is implicitly defined in the Appendix. Under risk neutrality, or if \( \lambda_1 = \lambda_2 \), then \( \rho_2(v) = 1 \) while under risk aversion \( \rho_2(v) > 1 \).

Proof: see Appendix.

The most interesting characteristic of Retractable English auctions is that risk aversion and affiliation now both operate in the same direction, enhancing the auctioneer's expected revenue.
This result stands in sharp contrast to either first price or English auctions for which we have shown that the two factors operate in opposite directions. Retractable English auctions clearly dominate non-retractable ones since the latters are constrained versions of the former ($\lambda_1=\lambda_2$). At the current stage of our investigations we have indications that Retractable English auctions also dominate first price auctions under most combinations of risk aversion and affiliation. For example, under constant risk aversion when $U(x) = x^\alpha$ for $0 < \alpha \leq 1$, the function $\rho_2$ is given by

$$\rho_2(v) = 1 + [\nu_2 \circ \theta(v) - 1] \cdot \left(1 - \frac{\psi(v)}{v}\right)$$  \hspace{1cm} (31)

where

$$\nu_2(x) = x^{\alpha-1} \cdot \frac{1 - x^{\alpha}}{1 - x^{\alpha}}$$

$$\theta(v) = \frac{v - \psi(v)}{v - \psi \circ \lambda^{-1}_1 \circ \lambda_2(v)}, \text{ so that } 0 \leq \theta(v) \leq 1$$  \hspace{1cm} (32)

The function $\nu_2$ has useful properties on $(0,1]$. Specifically,

$$\lim_{x \to 1} \nu_2(x) = \frac{1}{\alpha} \text{ and } \frac{d}{dx} \nu_2(x) < 0 \text{ on } (0,1)$$  \hspace{1cm} (33)

hence

$$\nu_2(x) > \frac{1}{\alpha} \text{ on } (0,1)$$  \hspace{1cm} (34)

However, the fact that $\nu_2 \circ \theta(v)$ in equation (31) is larger than $1/\alpha$ in equation (12) does not suffice to establish that $\rho_2(v)$ is larger than $\rho_1(v)$ since it might still be the case that first price auctions generate more shading by the bidders than Retractable English auctions. We currently believe that, even in the absence of affiliation, it should be possible to select the $\lambda$ functions in such a way that sufficient shading is generated for Retractable English auctions to dominate first price ones.\(^{16}\) In any event, the more affiliation is present, the more the comparison turns in favor of Retractable English auctions. This property makes Retractable English auctions especially attractive in the context of MW who essentially demonstrate that the policy of revealing information publicly, within the GS model, increases affiliation between bidders' valuations.

V. Refinements

In this section two issues are explored. First, we propose a randomization device which induces low-valued bidders to strictly prefer participation. Second, given that all bidders can observe the ascent of bids, we analyze the effect on strategies of this information.

\(^{16}\) A systematic investigation of this issue belongs to our research agenda but first requires the development of numerical techniques for the derivation of the bidders' and auctioneer's optimal strategies.
V.1 Low-Valued Bidders' Participation

It is critically important for the existence of our equilibrium strategies that low-valued bidders bid their valuations (or any invertible function thereof) even though under our current rules they have a zero expected surplus if \( v < x_\star \). Within the context of a Retractable English auction an incentive scheme is proposed below such that low-valued bidders strictly prefer participation.

When \( B < x_\star \), suppose there is a probability \( q(B) > 0 \) that the reserves \( \delta_1 \) and \( \delta_2 \) will be disabled, i.e. that the item will be awarded to the winning bidder at a price \( \psi(B) \).\(^{17}\) Since the need for incentives naturally decreases when \( B \) increases and vanishes once \( B > x_\star \) the auctioneer will typically set \( q(B) \) to be a decreasing function of \( B \) on \([0, x_\star]\) and to be zero beyond \( x_\star \).

Under this modification, a bidder with valuation \( v < x_\star \) may bid \( x < x_\star \) and still have a non-zero expected surplus which is given by\(^{18}\)

\[
\Pi(x, v) = \int_0^x q(w) \cdot U[v - \psi(w)] \cdot h(w|v) \, dw
\]  \hspace{1cm} (35)

The FOC for an equilibrium\(^{19}\) implies that

\[
\forall v < x_\star \quad q(v) > 0 \Rightarrow \psi(v) = v
\]  \hspace{1cm} (36)

When \( v \) and \( x \) are greater than \( x_\star \), the surplus function \( \Pi \) as defined in (23) is modified in two non-essential ways:

(i) First, it includes an additional factor of the form given in (35) where \( x \) is set at \( x_\star \). That factor does not depend on \( x \) and, hence, has no impact on the bid strategies. It is important since it prevents a bidder with valuation \( v \) (close to \( x_\star \)) from choosing \( x \leq x_\star \).

(ii) Second, it requires multiplying \( h(w|v) \) by the factor \([1-q(w)]\). The latter modification is easily dealt with in the context of Theorem 2. Its net impact is a moderate increase in shading by the bidders and a consequent reduction in the auctioneer's revenue.

The auctioneer then must determine \( q(b) \) on \([0, x_\star]\) in a way which achieves the optimal compromise between securing (at a cost) low valued bidders activity and maximizing revenue.

V.2 Observing the “Thermometer”

In our auctions every bidder can observe the bid sequence and yet this information is not useful to them. In this section we briefly investigate conditions under which the bid strategies would evolve with the progression of bids.

\(^{17}\) Reserves are not re-enabled if the last bidder bids above \( x_\star \). Otherwise, a discontinuity is created in the surplus at \( x_\star \) — bidders with valuations above but close to \( x_\star \) might find it profitable to bid below \( x_\star \).

\(^{18}\) In (35), \( W \in [0, x] \). Hence, \( x \) wins and \( W = B \).

\(^{19}\) Strictly speaking, the function \( q \) must be exogenously given to sustain a Nash equilibrium since for any \( q \) the auctioneer would always prefer a smaller one.
First, we consider an English auction. The expected surplus, as defined in (8), takes the simpler form

\[
\Pi(x, \nu) = \lambda(x) \int_0^\nu U[v - \delta \circ \psi(w)] \cdot h(w|\nu) \, dw \quad x \geq x_*
\]  

(37)

Though to this point we have treated \( \psi(x) \) as a precommitted bid, our thermometer implementation of the English auction allows our bidders to draw inferences from the mere ascent of the thermometer. Specifically, a bidder who is still active at some level \( x_0 \) and sees the thermometer ascending knows that someone (another bidder or the auctioneer) is also active. This information translates into the inequality \( \lambda(x_0) < \nu \). Conditionally on this information our typical bidder's expected surplus is given by

\[
\Pi(x, \nu) = \frac{1}{1 - H(\lambda(x_0)|\nu)} \lambda(x) \int_0^\nu U[v - \delta \circ \psi(w)] \cdot h(w|\nu) \, dw
\]  

(38)

which leaves the FOC unchanged. The information that \( \lambda(x_0) < \nu \) is useless to our bidders.

For a first price auction the situation changes dramatically. In such a case \( p_1' \neq 0 \). Conditionally on the information \( \lambda(x_0) < \nu \) the FOC would be of the form given in Theorem 1 except that the lower bound of the integral in (10) becomes \( \lambda(x_0) \) instead of zero. Hence, our bidder would revise his initial bid upwards.

The design of an optimal sequential bidding strategy for auctions with first price components \( p_1' \neq 0 \) when the bidder is allowed to see the clock running proves to be a formidable task — he exploits the information to reduce his expected payment by dynamically adjusting his bid in light of the information he collects on \( \nu \).

This explains why our thermometer implementation of the first price auction is designed in such a way that the price increases as long as at least one bidder stays active. Hence, active bidders collect no information on \( \nu \). This restriction is critically needed to validate our analysis as soon as \( p_1' \neq 0 \). Removal of it complicates the analysis. More importantly, its removal can only decrease the expected revenue of the auctioneer.

VI. Conclusion

In this paper, reserve pricing has been modeled as a function of the observed bid sequence. Reserve prices which are functions of observed bids generate higher expected revenue for the auctioneer than ex ante fixed reserves. The framework employed here not only integrates existing results in the literature (i.e MW and Riley and Samuelson (1981)) but extends them.

20 The reasoning that follows is the same for a Retractable English auction but requires additional notation.

21 The reason for not submitting initially a higher bid is that our bidder must account for the possibility that \( \nu \) might be less than \( \psi(x_0) \). In this case a high bid would excessively raise the price paid.
The extensions largely stem from our investigation of an ascending bid auction where the auctioneer can use two reserve prices, a non-retractable and a retractable one. The Retractable English auction, which parallels the observed practice of auctioneers at many ascending bid auctions, allows the exploitation of both affiliation and risk aversion. This last point is crucial given the schemes which have been analyzed in the literature —“pure” English auctions exploit affiliation but are incapable of taking advantage of risk aversion while first price auctions have the converse property.
Appendix

Proof of Lemma 1:

Since the last two equalities follow from the first we only establish that

\[ g(a, b) = n \cdot h(b|a) \cdot f(a) \quad (39) \]

First, let \( A, B, Y_3,...,Y_n \) be the order statistics for \((V_1,...,V_n)\). From MW equation (4) we have

\[
g(a, b) = n! \int_0^b \cdots \int_0^{y_{n-1}} f(a, b, y_3,...,y_n) \, dy_n \cdots dy_3 \quad a \geq b
\]

(40)

Instead, let \( B=V_1 \) and \( A,Y_3,...,Y_n \) be the order statistics for \((V_2,...,V_n)\).

\[
h(b|a) f(a) = (n-1)! \int_0^a \cdots \int_0^{y_{n-1}} f(b|a, y_3,...,y_n) f(a, y_3,...,y_n) \, dy_n \cdots dy_3
\]

(41)

for all pairs \((a, b)\)

The lemma follows from the last two equations. Note that by symmetry we can exchange the definitions of \( A \) and \( B \) in (41) and find also that \( g(a, b) = n \cdot h(a|b) \cdot f(b) \) for \( a \geq b \).

Proof of Theorem 1:

(i) Formula (10) immediately follows from the equilibrium condition

\[
\frac{\partial}{\partial x} \Pi(x, v) \bigg|_{x=v} = 0
\]

(42)

(ii) The integral in the r.h.s of (10) can be rewritten as

\[
U'[v - p(\psi(v), \psi(\zeta(v)))] \int_0^{\lambda(v)} \rho_1^l(\psi(v), \psi(w)) \cdot \psi'(v) \cdot h(w|v) \, dw
\]

(43)

with \( 0 < \zeta(v) < \lambda(v) \)

Under risk aversion and taking advantage of the fact that \( v \geq \psi(v) \geq \psi \circ \lambda(v) \) and hence \( v \geq p(\psi(v), \psi \circ \lambda(v)) \) we have successively

\[
\frac{U[v - p(\psi(v), \psi \circ \lambda(v))]}{U'[v - p(\psi(v), \psi(\zeta(v)))]} = \mu_1(v) \cdot \frac{U[v - p(\psi(v), \psi \circ \lambda(v))]}{U'[v - p(\psi(v), \psi \circ \lambda(v))]} = \rho_1(v) \cdot v - p(\psi(v), \psi \circ \lambda(v))
\]

(44)

17
where $\mu_i(v)$ and $\rho_1(v)$ are $\geq 1$. Hence the FOC implies for the existence of $\rho_1(v) \geq 1$ such that

$$
\rho_1(v) \cdot v \cdot h(\lambda(v)|v) \cdot \lambda'(v)
= p(\psi(v), \psi \circ \lambda(v)) \cdot h(\lambda(v)|v) \cdot \lambda'(v) + \int_0^{\lambda(v)} \int_0^t p(\psi(v), \psi(w)) \cdot \psi'(v) \cdot h(w|v) \, dw
$$

(45)

Our bidder's expected payment is

$$
P(v) = \int_0^{\lambda(v)} p(\psi(v), \psi(w)) \cdot h(w|v) \, dw \quad \text{for} \quad v > x_*
$$

(46)

Derivation of $P(v)$ together with condition (45) establishes formula (11). QED.

**Proof of Theorem 2:**

That $\psi(v) = v$ for $x_1^* < v < x_2^*$ and is given by equation (29) for $x_2^* < v$ immediately follows from the FOC

$$
\frac{\partial}{\partial x} \Pi(x, v) \big|_{x=v} = 0
$$

(47)

Continuity of $\psi$ at $x = x_2^*$ requires that

$$
\lim_{x \to x_2^*} \psi(x) = x_2^*
$$

(48)

from either side. Hence, from formula (29)

$$
x_2^* = \psi \circ \lambda^{-1} \lambda_2(x_*) = \psi \circ \lambda^{-1}(0) = \psi \left( x_1^* \right) = x_1^*
$$

(49)

wherefrom (i) follows.

The FOC and risk aversion implies the existence of $\nu_2(v) \geq 1$ such that

$$
\nu_2(v) \cdot [v - \psi(v)] \cdot h(\lambda_1(v)|v) \cdot \lambda'_1(v)
= \left[ \psi(v) - \psi \circ \lambda^{-1}_1 \circ \lambda_2(v) \right] \cdot h(\lambda_2(v)|v) \cdot \lambda'_2(v)
$$

(50)

$\rho_2(v) \geq 1$ is then defined by the identity

$$
\nu_2(v) \cdot [v - \psi(v)] = \rho_2(v) \cdot v - \psi(v)
$$

(51)
Our bidder's expected payment is

\[ P(v) = \int_0^{\lambda_1(v)} \delta_1 \circ \psi(w) \cdot h(w|v) \, dw + \int_0^{\lambda_2(v)} \left[ \delta_2 \circ \psi(w) - \delta_1 \circ \psi(w) \right] \cdot h(w|v) \, dw \]  

(52)

Equation (30) then follows by derivation and subsequent application of formulae (50) and (51). QED.

An Affiliated Distribution Example:

Let the distribution of \( V|\alpha \) be an inverted Weibull distribution (also known as a type 2 extreme value distribution) of the form

\[ F(v|\alpha) = \exp\left(-\alpha v^{-\beta}\right) \quad \alpha, \beta > 0 \]  

(53)

with known \( \beta \) and unknown \( \alpha \). See e.g. Graham, Marshall, and Richard (1987) for useful properties of that distribution. Let the distribution of \( \alpha \) be a gamma distribution with density function

\[ \nu(\alpha) = \frac{1}{\Gamma(\gamma)\theta^\gamma \alpha^{\gamma-1} \exp(-\alpha \theta)} \quad \gamma, \theta > 0 \]  

(54)

The distribution functions \( V \) and \( W|V \) are then given by

\[ F(v) = 1 - \left[ \frac{\theta}{\theta + v^{-\beta}} \right]^\gamma \] 

\[ H(w|v) = \left[ \frac{\theta + v^{-\beta}}{\theta + v^{-\beta} + (n-1)w^{-\beta}} \right]^{\gamma+1} \]  

(55)

It can be verified that the affiliation condition (3) holds for all \((\beta, \theta, \gamma)\). The correlation between \( V \) and \( W \) is given by

\[ \rho = \frac{\left\{ \frac{\Gamma(\gamma+\frac{1}{\beta})}{\Gamma(\gamma)\gamma^{\frac{1}{\beta}}} - \left[ \frac{\Gamma(\gamma+\frac{1}{\beta})}{\Gamma(\gamma)\gamma^{\frac{1}{\beta}}} \right]^2 \right\}}{\left\{ \frac{\Gamma(1-\frac{1}{\beta})\Gamma(\gamma+\frac{1}{\beta})}{\Gamma^2(1-\frac{1}{\beta})\Gamma(\gamma)\gamma^{\frac{1}{\beta}}} - \left[ \frac{\Gamma(\gamma+\frac{1}{\beta})}{\Gamma(\gamma)\gamma^{\frac{1}{\beta}}} \right]^2 \right\}} \]  

(56)

Typical values are:

\[ (i) \quad \rho = .24 \text{ for } \beta = 3 \text{ and } \gamma = .8 \] 

\[ (ii) \quad \rho = .91 \text{ for } \beta = 8 \text{ and } \gamma = .1 \]  

(57)
Bibliography


